Independent Dominating Sets of Lexicographic Product Graphs of Cayley Graphs with Arithmetic Graphs

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Abstract:

Domination in graphs is the fast growing area in Graph Theory that has emerged rapidly in the last five decades. Domination in graphs has applications to several fields such as facility location problems, School Bus Routing, Computer Communication Networks, Radio Stations, Locating Radar Stations Problem etc. Every branch of mathematics employs some notion of a product that enables the combination or decomposition of its elemental structures. Product of graphs are introduced in graph theory very recently and developing rapidly. An Euler totient Cayley graph is an arithmetic graph constructed using the Euler totient function. In this paper, we consider Lexicographic product graphs of Cayley graphs with Arithmetic graphs and discuss independent domination number of these graphs.
Keywords: Dominating Set, Independent Dominating Set, Lexicographic Product Graph, Euler Totient Cayley Graph, Arithmetic V_n Graph.

I. Introduction

The theory of domination was formalized by Berge [1] and Ore [2] in 1962. Since then it has developed rapidly and various variations of domination are introduced and studied. The independent domination number and the notation $i(G)$were introduced by Cockayne and Hedetniemi in [3, 4] and later developed by Allan and Laskar [5]. Independent dominating sets have been studied extensively in the literature [6, 7].

A dominating set D of a graph G is a subset of vertex set V of G such that every vertex in $V - D$ is adjacent to at least one vertex in D. The minimum cardinality of a dominating set D of G is called the domination number of G and is denoted by $\gamma(G)$.

A subset of vertices of V of a graph G is called an independent set if no two vertices in it are adjacent. An independent dominating set of $G$is a set that is both dominating and independent in G. The independent domination number of G, denoted by $\gamma_i(G)$, is the minimum cardinality of an independent dominating set.

Lexicographic Product Graph $G_1 o G_2$

The lexicographic product was first studied by Felix Hausdorff in the year 1914. Later this product was introduced as the composition of graphs by Harary in the year 1959. There has been a rapid growth of research on the structure of this product and their algebraic settings, after the publication of the paper, on the group of the composition of two graph by Haray, F [8]. Geller, D and Stahl [9] determined the chromatic number and other functions of this product in the year 1975. Feigenbaum and Schaffer [10] carried their research on the problem of recognizing whether a graph is a lexicographic product is equivalent to the graph isomorphism problem in the year 1986. Imrich and Klavzar [11] discussed the automorphisms, factorizations and non-uniqueness of this product.

This product is in general non-commutative. But two graphs G and H commute with respect to the lexicographic product if G and H are complete or if both are totally disconnected graphs.

We know that if G_1 and G_2 are two simple graphs with their vertex sets as $V_1 = \{u_1, u_2, ..., u_l\}$ and $V_2 = \{v_1, v_2, ..., v_m\}$ respectively then the lexicographic product of these two graphs denoted by $G_1 O G_2$ is defined as the graph with vertex set $V_1 \times V_2$, where $V_1 \times V_2$ is the Cartesian product of the sets V_1 and V_2 and any two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 O G_2$ are adjacent if

(i) $u_1 u_2 \in E(G_1)$ or

(ii) $u_1 = u_2$ and $v_1 v_2 \in E(G_2)$.

Euler Totient Cayley Graph $G(Z_n, \varphi)$

Madhavi [12] introduced the concept of Euler totient Cayley graphs and studied some of its properties. She gave methods of enumeration of disjoint Hamilton cycles and triangles in these graphs.

For each positive integer n, let Z_n be the additive group of integers modulo n and let S be the set of all integers less than n and relatively prime to n. The Euler totient Cayley graph $G(Z_n, \varphi)$ is defined as the graph whose vertex set V is given by $Z_n = \{0,1,2, ... n-1\}$ and the edge set is $E = \{(x,y)/x - y \in S \text{ or } y - x \in S\}$.

Clearly as proved in [12], the Euler totient Cayley graph $G(Z_n, \varphi)$ is

1. a connected, simple and undirected graph,
2. $\varphi(n)$ - regular and has $\frac{n.\varphi(n)}{2}$ edges,
3. Hamiltonian,
4. Eulerian for $n \geq 3$,
5. bipartite if n is even and
6. complete graph if n is a prime.

The independent domination number of these graphs are studied by the authors [13] and the following results are required and they are presented without proofs.

Theorem 1.1: If n is a prime, then the independent domination number of $G(Z_n, \varphi)$ is 1.

Theorem 1.2: The independent domination number of $G(Z_n, \varphi)$ is 2, if $n = 2p$ where p is an odd prime.

Theorem 1.3: Suppose n is neither a prime nor $2p$. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots\dots p_k^{\alpha_k}$, where p_1, p_2, … p_k are primes and $\alpha_1, \alpha_2, ... \alpha_k$ are integers ≥ 1, then the independent domination number of $G(Z_n, \varphi)$ is $\frac{n}{p_k}$.

Arithmetic V_n Graph

Vasumathi and Vangipuram [14] introduced the concept of Arithmetic V_n graphs and studied some of its properties.

Let n be a positive integer such that $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots\dots p_k^{\alpha_k}$. Then the Arithmetic V_n graph is defined as the graph whose vertex set consists of the divisors of n and two vertices u, v are adjacent in V_n graph if and only if GCD $(u, v) = p_i$, for some prime divisor p_i of n.

In this graph vertex 1 becomes an isolated vertex. Hence we consider Arithmetic graph V_n without vertex 1 as the contribution of this isolated vertex is nothing when the properties of these graphs and enumeration of some domination parameters are studied.

Clearly, V_n graph is a connected graph. Because when n is a prime, V_n graph consists of a single vertex. Hence it is a connected graph. In other cases, by the definition of adjacency in V_n, there exist edges between prime number vertices and their prime power vertices and also to their prime product vertices. Therefore each vertex of V_n is connected to some vertex in V_n.

The independent domination number of these graphs are obtained by the authors and the proof of the following theorem can be found in [13].

Theorem 1.4: If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots\ldots p_k^{\alpha_k}$, where $p_1, p_2, \ldots p_k$ are primes and $\alpha_1, \alpha_2, \ldots . \alpha_k$ are integers ≥ 1, then the domination number of $G(V_n)$ is given by

$$\gamma_i(G(V_n)) = \begin{cases} k-1 & \text{if } \alpha_i = 1 \text{ for more than one i} \\ k & \text{Otherwise.} \end{cases}$$

where k is the core of n.

II. INDEPENDENT DOMINATION IN LEXICO- GRAPHIC PRODUCT GRAPH $G_1 \circ G_2$

Let G_1 denote $G(Z_n, \varphi)$ graph and G_2 denote $G(V_n)$ graph. Then G_1 and G_2 are simple graphs as they have no loops and multiple edges. Hence by the definition of lexicographic product, $G_1 \circ G_2$ is also a simple graph.

Now we discuss independent dominating sets of Lexicographic product graphs $G_1 \circ G_2$.

Theorem 2.1: If $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots\ldots p_k^{\alpha_k}$, where $\alpha_i > 0$, then the independent domination number of $G_1 \circ G_2$ is given by

$$\gamma_i(G_1 \circ G_2) = \gamma_i(G_1)\, \gamma_i(G_2)$$

$$= \begin{cases} \dfrac{n}{p_k} \cdot (k-1), & \text{if } \alpha_i = 1 \text{ for more than one i} \\ \dfrac{n}{p_k} \cdot k, & \text{Otherwise.} \end{cases}$$

where k is the core of n.

Proof: Let V_1 , V_2 and V denote the vertex sets of the graphs G_1 , G_2 and $G_1 \circ G_2$ respectively. By Theorem 1.3 in [3], it is clear that the elements of the set V_1 can be divided into disjoint sets $D_i = \left\{ r.p_k + i \;/\; r = 0, 1, 2, \ldots\ldots, \left(\frac{n}{p_k} - 1\right) \right\}$

for $1 \leq i \leq p_k$, such that each D_i is an independent dominating set of G_1 with minimum cardinality $\frac{n}{p_k}$.

Inparticular, let $D_{i_1} = \left\{ 0,\ p_k,\ 2p_k,\ \ldots\ldots,\ \left(\frac{n}{p_k} - 1\right) p_k \right\}$ be an independent dominating set of G_1 with minimum cardinality $\frac{n}{p_k}$.

By Theorem 1.4 in [13], it is clear that

$$\gamma_i(G_2) = \begin{cases} k-1, & \text{if } \alpha_i = 1 \text{ for more than one i} \\ k, & \text{Otherwise} \end{cases}$$

Further we know that if $\alpha_i > 1$, for all i or $\alpha_i = 1$ for only one i, then $D_{i_2} = \{ p_1, p_2, \ldots\ldots, p_k \}$ is an independent dominating set of G_2 with minimum cardinality k, and

if $\alpha_i - 1$ for more than one i, then the set $D_{i'_2} = \{ p_1, p_2, \ldots\ldots, p_{i-2}, p_{i-1} \cdot p_i, p_{i-1}, \ldots\ldots, p_k \}$ is an independent dominating set of G_2 with minimum cardinality $k-1$. In order to find an independent dominating set of $G_1 \circ G_2$ we proceed in two cases:

Case 1: Suppose $\alpha_i > 1$ for all i or $\alpha_i = 1$ for only one i. Consider $D = D_{i_1} \times D_{i_2} = \left\{ 0, p_k, 2p_k, 3p_k, \ldots\ldots, \left(\frac{n}{p_k} - 1\right) p_k \right\} \times \{ p_1, p_2, \ldots\ldots, p_k \}$

$$- \{ (0, p_1), (0, p_2), (0, p_3), \ldots (0, p_k),$$
$$(p_k, p_1), (p_k, p_2), (p_k, p_3), \ldots (p_k, p_k) \ldots\ldots\ldots\ldots$$
$$\left(\left(\frac{n}{p_k} - 1\right) p_k, p_1\right), \left(\left(\frac{n}{p_k} - 1\right) p_k, p_2\right), \left(\left(\frac{n}{p_k} - 1\right) p_k, p_3\right), , \left(\left(\frac{n}{p_k} - 1\right) p_k, p_k\right) \}$$

Suppose $x, y \in D$ such that $x = (ap_k,\ p_i)$, $y = (bp_k,\ p_j)$
where $0 \le a, b \le \frac{n}{p_k} - 1$, $1 \le i, j \le k$.

Vertices ap_k, bp_k are not adjacent because GCD $(ap_k - bp_k, n)$ =GCD $((a-b)p_k, n) \neq 1$. Further if $ap_k = bp_k$, then p_i, p_j are not adjacent because GCD $(p_i, p_j) = 1$ for $i \neq j$. Hence by the definition of lexicographic product x and y are not adjacent for $x, y \in D$. Thus D becomes an independent set of $G_1 o G_2$.

We now claim that D is a dominating set of $G_1 o G_2$. Let $(u,v) \in V - D$. Since $D = D_{i_1} \times D_{i_2}$, it follows that $u \in D_{i_1}$ and $v \notin D_{i_2}$ or $u \notin D_{i_1}$ and $v \in D_{i_2}$ or $u \notin D_{i_1}$ and $v \notin D_{i_2}$.

Therefore the following sub-cases arise.

Subcase 1: Suppose $u \in D_{i_1}$ and $v \notin D_{i_2}$.

Since D_{i_2} is a dominating set of G_2, for $v \notin D_{i_2}$, certainly there exists atleast one vertex in $\{p_1, p_2, \ldots, p_k\}$ say p_j such that v and p_j are adjacent to each other. Then by the definition of lexicographic product, (u,v) in $V - D$ is adjacent to (u, p_j) in D.

Subcase 2: Suppose $u \notin D_{i_1}$ and $v \in D_{i_2}$ or $u \notin D_{i_1}$ and $v \notin D_{i_2}$.

Since $u \notin D_{i_1}$, u is adjacent to at least one vertex cp_k, for $0 \le c \le \frac{n}{p_k} - 1$, in D_{i_1} as D_{i_1} is a dominating set of G_1. Hence by the definition of lexicographic product, in either of the cases $v \in D_{i_2}$ and $v \notin D_{i_2}$, we have (u,v) in $V - D$ is adjacent to $(cp_k,\ v_l)$, $\forall\ v_l \in V_2$. In particular (u,v) in $V - D$ is adjacent to $(cp_k, p_1), (cp_k, p_2), \ldots\ldots, (cp_k, p_k)$ in D.

Thus we conclude that every vertex (u,v) in $V - D$ is adjacent to at least one vertex in D.

We now prove that D is minimum. Suppose we remove a vertex (dp_k, p_l) from D. where $0 < d < \frac{n}{p_k} - 1$ and $1 \le l \le k$. Then the deleted vertex (dp_k, p_l) is in $V - D$. Further this vertex (dp_k, p_l) is not adjacent to any vertex (sp_k, p_j) of D for $0 \le s \le \frac{n}{p_k} - 1$. This is because for $0 \le d, s \le \frac{n}{p_k} - 1$, GCD $(dp_k - sp_k, n) \neq 1$. Hence by the definition of adjacency in G_1, vertex dp_k is not adjacent to vertex sp_k. Further if $dp_k = sp_k$, then by the definition of adjacency in G_2 vertex p_l is not adjacent to vertex p_j as GCD $(p_l, p_j) = 1$ for $l \neq j$. Hence by the definition of lexicographic product, vertex (dp_k, p_l) is not adjacent to any vertex (sp_k, p_j) of D. Thus $D - \{(dp_k, p_l)\}$ is not a dominating set. Thus D becomes minimal with cardinality $\frac{n}{p_k} \cdot k$. If a minimal dominating set is formed in any other way, then the cardinality of such a set is not smaller than that of D. This follows from the properties of the prime divisors of a number.

Thus D becomes an independent dominating set with minimum cardinality $\frac{n}{p_k} \cdot k$.

Case 2: Suppose $\alpha_i = 1$ for more than one i. Consider $D' = D_{i_1} \times D_{i_2'}$

$= \left\{0, p_k, 2p_k, 3p_k, \ldots, (\frac{n}{p_k} - 1)p_k\right\} \times \{p_1, p_2, \ldots, p_{i-2}, p_{i-1} \cdot p_i, p_{i+1}, \ldots, p_k\}$

Proceeding as in Case 1, we can show that D' is an independent dominating set of $G_1 o G_2$ with minimum cardinality $\frac{n}{p_k} \cdot (k-1)$.

Therefore $\gamma_i(G_1 o G_2) = |D'| = \frac{n}{p_k} \cdot (k-1)$ ■

2. ILLUSTRATIONS

$n = 4$

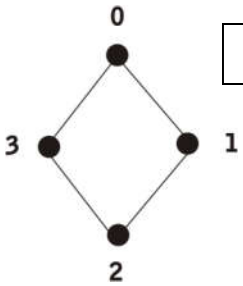


Fig 1: $G_1 - G(Z_4, \varphi)$

Fig 2: $G_2 = G(V_4)$

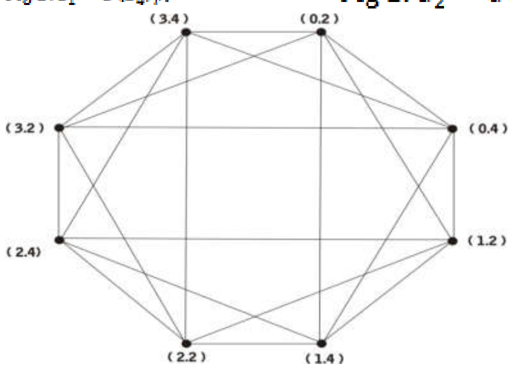
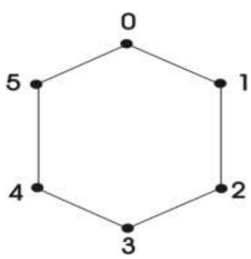


Fig 3: $G_1 \circ G_2$

Minimum Independent Dominating set:

{(0,2), (2,2)} and $\gamma_i = 2$

$n = 6$



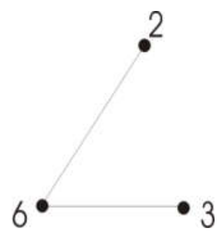


Fig 4: $G_1 = G(Z_6, \varphi)$

Fig 5: $G_2 = G(V_6)$

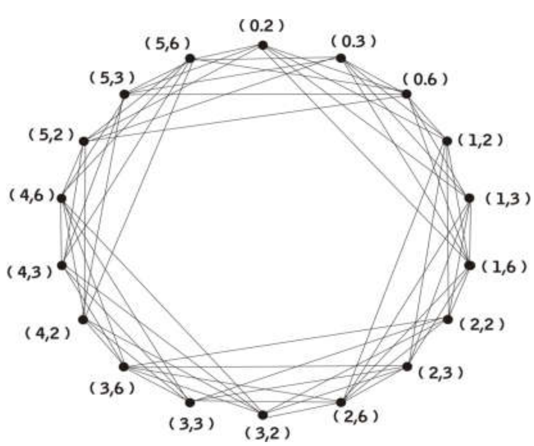


Fig 6: $G_1 \circ G_2$

Minimum Independent Dominating set:

{(0,6), (3,6)} and $\gamma_i = 2$

$n = 11$

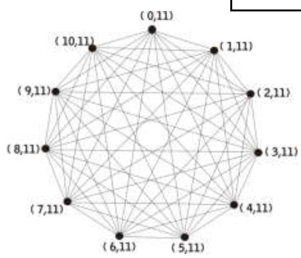


Fig 7: $G_1 = G(Z_{11}, \varphi)$ Fig 8: $G_2 = G(V_{11})$

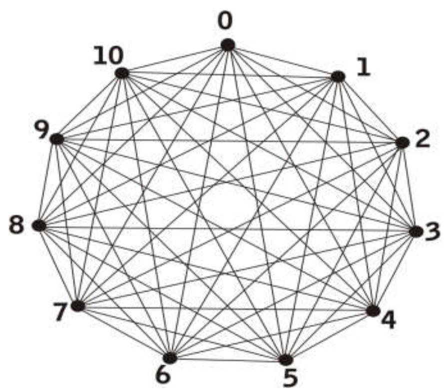


Fig 9: $G_1 \cup G_2$

Minimum Independent Dominating set:

{(0,11)} and $\gamma_i = 1$

4. CONCLUSION

Graph Theory is young but rapidly maturing subject. Its basic concepts are simple and can from many different subjects. The purpose of this work is to familiarize the reader with the Cartesian Cayley graph with Arithmetic Vn graph. It is useful other Researchers for further studies of other d product graphs and their relevance in both combinatorial problems and classical algebraic problems.

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