A STUDY OF FOURIER TRANSFORM PAIR: FREQUENCY AND TIME DOMAIN

Research Guide

Manjunatha H, **Dr. K. M Nagaraja**, DEPT.OF MATHEMATICS

Research Scholar, **J.S.S Academy for Technical Education,**

DEPT.OF MATHEMATICS,VTU **Bangalore**

EMAIL:MANJUMSC82@GMAIL.COM

ABSTRACT: THIS PAERCONTAINS the amplitude A and Phase angle θ for $C_k$are 0.59272353, and 2.13770783 radians, respectively, Derivations of DFT Formulas, Non-Periodic Function, Correctly reconstructed signal, Funciton to be sampled and "Windowing" Sample Problem

---

1.1 Introduction:

In Chapter 2, Fourier approximations were expressed in the time domain. The amplitude (vertical axis) of a given periodic function can be plotted versus time (horizontal axis), but it can also be plotted in the frequency domain (1.1-1.6) as shown in Figure 1.

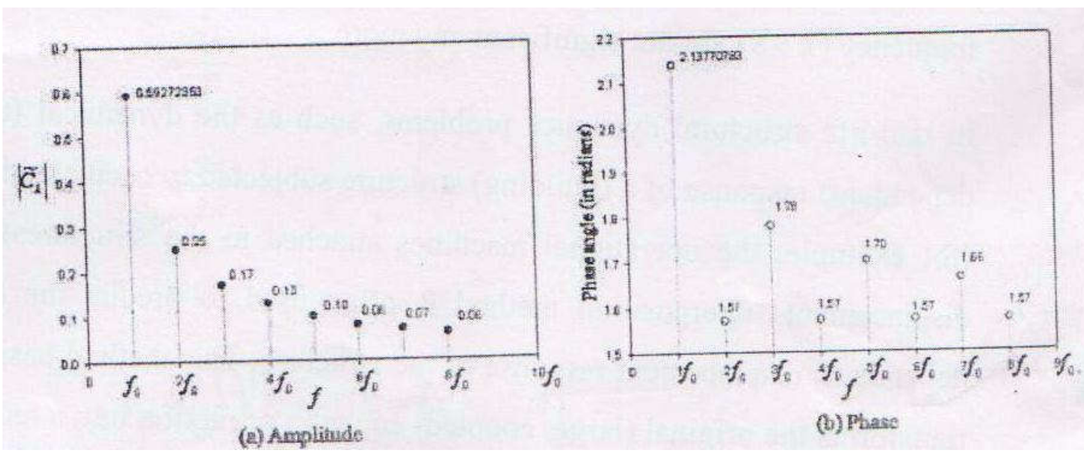


Figure 1 Periodic Function in Frequency

The advantages of plotting the amplitude of a given periodic function in frequency domain (instead of time domain) are due to the following reasons:

For a specific value "k" (say k=2) of the Fourier series in the time domain, one has to plot the entire curve to observe the amplitude of a given periodic function (recall $f_2(t) = a_0 + a_1\cos(t)+b_1\sin(t)+a_2\cos(2t)+b_2\sin(2t)$, see Example 1 in Chapter 2). However, in the frequency domain, the amplitude can be plotted as a single point (see Figure 1a).

In the frequency domain, one can easily identify which frequency(or corresponding to which value of "k") contributes the most to the amplitude (see Figure 1(a)) , where such information is not readily available if time domain is used.

From the amplitude plot in frequency domain (see Figure 1(a)), one can easily identify that contributions to the amplitude beyond the 8^{th} frequency (k>8) are not significant any more.

In real –life structural dynamics problems, such as the dynamical(time-dependent) response of a (building) structure subjected to oscillated loads (for example , the operational machines attached to the structures), the displacement superposition method is often used to predict the (time dependent) displacement of the structure. This method basically transforms the original (large, coupled) equation of motion into a reduced (much smaller size, un-coupled) equation of motion by making use of the few free vibration mode shapes and its associated frequencies.

Knowledge of which frequencies (and the corresponding mode shapes) that have the most contribution to the predicted dynamical response(such as nodal displacement response) plays crucial roles for the algorithms ' efficiencies.

Detailed explanations on how to obtain Figure 1(a), and 1(b) are now presented in the following sections.

Explanation of Figure 1(a) and 1(b)

One starts with Equation (1.18) and (1.20) of Chapter 2 f(t) = $\sum_{k=-\infty}^{\infty} C_k e^{ikw_0 t}$

where,

$$C_k = \left(\frac{1}{T}\right)\left\{\int_0^T f(t)\times e^{-ikw_0 t}dt\right\}$$

For the periodic function shown in Example 1 of Chapter 2 (or Figure 1 of Chapter 2), one has

$$w_0 = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$$C_k = \left(\frac{1}{T}\right)\left\{\int_0^T t\times e^{-ikw_0 t}dt + \int_0^{2\pi} \pi\times e^{-ikt}dt\right\}$$

Define, and using "integration by parts" formula

$$A = \int_0^{\pi} t\times e^{-ikt}dt = \left[t\times\left(\frac{-1}{ik}\right)e^{-ikt}\right]_0^{\pi} + \int_0^{\pi}\left(\frac{1}{ik}\right)^{-ikt} dt$$

$$A = \left[\left(\frac{-\pi}{ik}\right)e^{-ik\pi}\right]_0^{\pi} + \left(\frac{1}{ik}\right)\left[\left(-\frac{1}{ik}\right)e^{-ikt}\right]_0^{\pi}$$

$$= \left[\left(\frac{-\pi}{ik}\right)e^{-ik\pi}\right] + \left(\frac{1}{k^2}\right)\left[e^{-ik\pi} - 1\right]$$

$$= \left[\left(\left(\frac{-\pi}{ik}\right)e^{-ik\pi} + \left(\frac{1}{k^2}\right)e^{-ik\pi} - \frac{1}{k^2}\right)\right]$$

$$B = \pi\int_0^{2\pi} e^{-ikt}dt = \left[\left(e^{-ikt}\right)\left(\frac{-\pi}{ik}\right)\right]_{\pi}^{2\pi}$$

$$= \left(\frac{-\pi}{ik}\right)\left[e^{-ik2\pi} - e^{-ik\pi}\right]$$

$$= \left(\frac{\pi i}{k}\right)\left[e^{-ik2\pi} - e^{-ik\pi}\right]$$

Thus,

$$C_k = \left(\frac{1}{2\pi}\right)\{A + B\}$$

$$= \left(\frac{1}{2\pi}\right)\left\{e^{-ik\pi}\left(\frac{\pi i}{k} + \frac{1}{k^2} - \frac{\pi i}{k}\right) - \frac{1}{k^2} + \left(\frac{\pi i}{k}\right)e^{-ik2\pi}\right\}$$

Using the following Euler identities

$$e^{-ik\pi} = \cos(-k\pi) + i\sin(-k\pi) = \cos(k\pi) - i\sin(k\pi) = \cos(k\pi)$$

$$e^{-ik(2\pi)} = \cos(k(2\pi)) - i\sin(k(2\pi)) = \cos(k(2\pi))$$

$$C_k = \left(\frac{1}{2\pi}\right)\left\{\cos(k\pi)\times\left(\frac{1}{k^2}\right) - \frac{1}{k^2} + \left(\frac{\pi i}{k}\right)\cos(k2\pi)\right\}$$

or,

$$C_k = \left(\frac{1}{2\pi}\right)\left\{\left(\frac{1}{k^2}\right)\cos(k\pi) - \frac{1}{k^2} + \left(\frac{\pi i}{k}\right)\right\}$$

Also, since:

$$\cos(k\pi) = \begin{cases} -1 \; for \; k = odd \; integer (= 1,3,5,7 \ldots \ldots) \\ +1 \; for \; k = even \; integer (= 2,4,6,8 \ldots .) \end{cases}$$

Hence:

$$\cos(k\pi) = (-1)^k$$

Thus,

$$C_k = \left(\frac{1}{2\pi}\right)\left\{\frac{(-1)^k}{k^2} - \frac{1}{k^2} + \frac{\pi i}{k}\right\}$$

$$C_k = \left(\frac{1}{2\pi k^2}\right)\left[(-1)^k - 1\right] + \left(\frac{1}{2k}\right)i$$

From equation (1.15) in Chapter 2, one has:

$$C_k = \frac{a_k - ib_k}{2}$$

Hence upon comparing the above 2 equations, one concludes

$$a_k = \left(\frac{1}{\pi k^2}\right)\left[(-1)^k - 1\right]$$

$$b_k = \left(\frac{-1}{k}\right)$$

Remarks:

For k = 1,2,3,4….8; the values of $a_k$and b_k (based on the above 2(formulas) are exactly identical as the ones presented earlier in Example 1 in Chapter 2.

Thus,

$$C_1 = \frac{a_1 - ib_1}{2}$$

$$= \frac{\frac{-2}{\pi} - i(-1)}{2}$$

$$= \frac{-1}{\pi} + \frac{1}{2}i$$

$$C_2 = \frac{a_2 - ib_2}{2}$$

$$= \frac{0 - i(\frac{-1}{2})}{2}$$

$$= 0 + \tfrac{1}{4} i$$

$$C_3 = \frac{a_3 - ib_3}{2}$$

$$= \frac{\frac{-2}{9\pi} - i(\frac{-1}{3})}{2}$$

$$= \left(\frac{-1}{9\pi}\right) + \frac{1}{6}i$$

$$C_4 = \frac{a_4 - ib_4}{2}$$

$$= \frac{0 - i(\frac{-1}{4})}{2}$$

$=0+\frac{1}{8}i$

$C_5 \quad = \quad \frac{a_5 - ib_5}{2}$

$= \quad \frac{\frac{-2}{25\pi} - i(\frac{-1}{5})}{2}$

$= \quad \left(\frac{-1}{25\pi}\right) + \frac{1}{10}i$

$C_6 \quad = \quad \frac{a_6 - ib_6}{2}$

$= \quad \frac{0 - i(\frac{-1}{6})}{2}$

$= \quad 0 + \frac{1}{12}i$

$C_7 \quad = \quad \frac{a_7 - ib_7}{2}$

$= \quad \frac{\frac{-2}{49\pi} - i(\frac{-1}{7})}{2}$

$= \quad \left(\frac{-1}{49\pi}\right) + \frac{1}{14}i$

$C_8 \quad = \quad \frac{a_8 - ib_8}{2}$

$= \quad \frac{0 - i(\frac{-1}{8})}{2}$

$= \quad 0 + \frac{1}{16}i$

In general, one has

$$C_k = \begin{cases} \frac{-1}{k^2\pi} + \left(\frac{1}{2k}\right) i for\ k = 1,3,5,7 \ldots\ldots = odd\ integer \\ \left(\frac{1}{2k}\right) i\ for\ k = 2,4,6,8 \ldots\ldots = even\ integer \end{cases}$$

Representation of a complex number in polar coordinates

In Cartesian (rectangular) coordinates, a complex number C_k can be expressed as:

$C_k \quad = \quad R_k + (I_k)i$

Where R_k and I_k represents the real and imaginary components of C_k , respectively.

In polar coordinates, a complex number C_k can be expressed as:

$C_k \quad = \quad Ae^{i\theta} = A\{\cos(\theta) + i\sin(\theta) = \{A\cos(\theta)\} + \{A\sin(\theta)\}i$

Where A and θ represents the amplitude and phase angle of C_k respectively (see Figure 2)

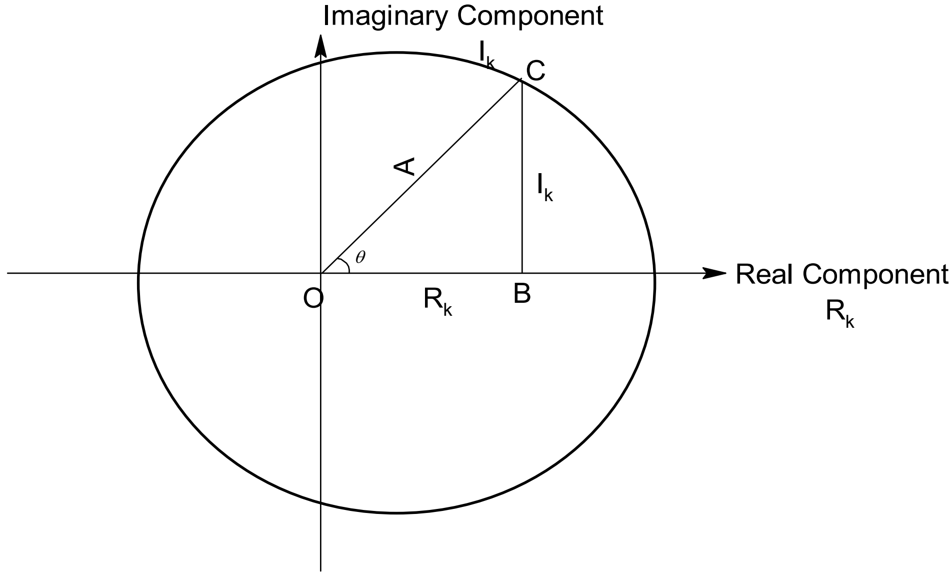


Figure 2 Representation of a complex number in polar coordinates

Thus, one obtains the following relations between the cartesian and polar coordinate systems:

$$R_k = A\cos(\theta)$$

$$I_k = A\sin(\theta)$$

Hence:

$$R_k^2+I_k^2 = A^2\cos^2(\theta)+A^2\sin^2(\theta) = A^2[\cos^2(\theta)+\sin^2(\theta)$$

$$A^2 = R_k^2+I_k^2$$

$$A = \sqrt{R_k^2 + I_k^2}$$

$\text{Cos}(\theta)=\frac{R_k}{A}$ implies $\theta=\cos^{-1}\left(\frac{R_k}{A}\right)$

$\sin(\theta)=\frac{I_k}{A}$ implies $\theta=\sin^{-1}\left(\frac{I_k}{A}\right)$

Based on the above 3 formulas, the complex numbers C_k , for k = 1,2,3……….,8 can be expressed as

$$C_1 = \frac{-1}{\pi}+\left(\frac{1}{2}\right)i = (0.59272353)e^{i(2.13770783)}$$

Hence, the amplitude A and Phase angle θ for $C_k$are 0.59272353, and 2.13770783 radians, respectively. The readers should refer to Figures 1(a) and 1(b) to confirm the plotted values

Important Notes:

If one uses the formula

$$\theta = \cos^{-1}\left(\frac{R_k}{A}\right)$$

$$= \cos^{-1}\left(\frac{\frac{-1}{\pi}}{0.59272353}\right)$$

$$= 2.13770783 \text{ radians}$$

$$= 122.48^0$$

However, the other formula for θ gives:

$$\theta = \sin^{-1}\left(\frac{I_k}{A}\right)$$

$$= \sin^{-1}\left(\frac{0.5}{0.59272353}\right)$$

$$= 1.0038848 \text{ radians}$$

$$= 57.52'$$

since R_k is negative, and I_k is positive, the angle θ must be in the 2nd (or upper left) quadrant of a circle (or $90^0 \leq \theta \leq 180^0$) Thus, the correct value for θshould be 2.13770783 radians (or 122.48^0) and the other value for θ=1.0038848 radians must be discarded.

Similarly, one obtains

$$C_2 = 0+\frac{1}{4}i$$

$$= (0.25)e^{i\left(\frac{\pi}{2}\right)}$$

$$= (0.25)e^{i(1.5707963)}$$

$$C_3 = \left(\frac{-1}{9\pi}\right)+\frac{1}{6}i$$

$$= (0.17037798)e^{i(1.7799009)}$$

$$C_4 = 0+\frac{1}{8}i$$

$$= (0.125)e^{i\left(\frac{\pi}{2}\right)}$$

$$= (0.125)e^{i(1.5707963)}$$

$$C_5 = \left(\frac{-1}{25\pi}\right)+\frac{1}{10}i$$

$$= (0.100807311)e^{i(1.69743886)}$$

$$C_6 = 0+\frac{1}{12}i$$

$$(0.08333333)e^{i(\frac{\pi}{2})}$$

$$= (0.08333333)e^{i(1.5707963)}$$

$$C_7 = \left(\frac{-1}{49\pi}\right)+\frac{1}{14}i$$

$$=(.07172336)e^{i(1.6614925)}$$

$$C_8 = 0+\frac{1}{16}i$$

$= (0.0625)e^{i(\frac{\pi}{2})}$

In summary, the given periodic function (shown in Example 1 of Chapter 2) can also be expressed in complex number formats, in polar coordinated with the amplitudes and phase angles given in the following table (also refer to Figures 1(a) and 1(b)).
Table 1 Amplitude and phase angle (in radians) for varying k values.

K	Amplitude	Phase Angle(radians)
1	0.59272353	2.13770783
2	0.25	$\frac{\pi}{2} = 1.57079633$
3	0.14037798	1.77990097
4	0.125	$\frac{\pi}{2}$
5	0.100807311	1.69743886
6	0.8333333	$\frac{\pi}{2}$
7	0.7172336	1.66149251
8	0.0625	$\frac{\pi}{2}$

1.2 Non-Periodic Function
Recall that a periodic function can be expressed in terms of the exponential form, according to Equations (18,20) of Chapter 1 as

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{ikw_0 t}$$

$$C_k = \left(\frac{1}{T}\right)\left\{\int_0^T f(t) \times e^{-ikw_0 t} dt\right\}$$

Define the following function

$$F(ikw_0) = \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-ikw_0 t} dt \qquad (1.1)$$

Where $F(ikw_0)$ is a function of I,k, and w_0
The equation (2.20) of Chapter 2 can be written as

$$C_k = \left(\frac{1}{T}\right) \times F(ikw_0) \qquad (1.2)$$

and equation (2.18) of Chapter 2 becomes

$$f(t) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{T}\right) \times F(ikw_0) e^{ikw_0 t} \qquad (1.3)$$

A non-periodic function f_{np} can be considered as a periodic function, with the period

$T \rightarrow \infty$, or $\Delta F = \frac{1}{T} \rightarrow 0$, (see Figure3)

From equations (2.6) and (2.7) from Chapter 2, one gets

$$w_0 = 2\pi f \qquad (1.4)$$

$$= \frac{2\pi}{T}$$

$$= 2\pi(\Delta f)$$

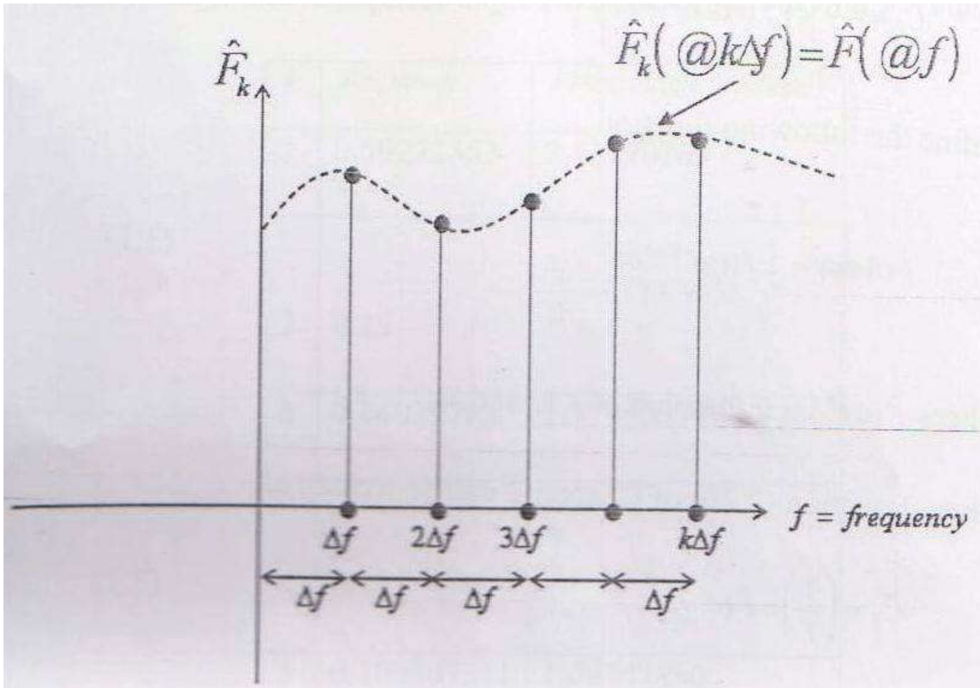


Figure 3 Discretization of frequency data.
From equation (3.3), one obtains

$$f_{np}(t) = \lim_{\substack{T\to\infty \\ \text{or } \Delta f\to 0}} f(t)$$

$$= \lim_{\Delta f\to 0} \sum_{k=-\infty}^{\infty} (\Delta f)\times F(ikw_0)e^{ikw_0 t} \qquad (1.5)$$

In the above equation, the subscript “np” denotes non-periodic function.

$$f_{np}(t) = \lim_{\Delta f\to 0} \sum_{k=-\infty}^{\infty} (\Delta f)\times F(ik2\pi\Delta f)e^{ik2\pi\Delta f t} \qquad (1.6)$$

Realizing that kΔf=f (See Figure 3) , the above equation becomes

$$f_{np}(t) = \int df \times F(i2\pi f)e^{i2\pi ft}$$

$$f_{np}(t) = \int F(i2\pi f)e^{i2\pi ft}df \qquad (1.7)$$

Multiplying and dividing the right-hand side of the equation by 2π, one obtains

$$f_{np}(t) = \left(\frac{1}{2\pi}\right)\int F(i2\pi f)e^{i2\pi ft}d(2\pi f)$$

$$=\left(\frac{1}{2\pi}\right)\int_{-\infty}^{\infty} F(iw_0)e^{iw_0 t}d(w_0) \text{ ; inverse Fourier transform} \qquad (1.8)$$

Using the definition stated in Equation (1) , one has

$$F(iw_0) = \int_{-\infty}^{\infty} f_{np}(t)e^{-iw_0 t}d(t) \text{ ; Fourier transform} \qquad (1.9)$$

Thus, equations (3.9) and (3.8) will transform a non-periodic function from time domain to frequency domain, and from frequency domain to time domain, respectively.

3.3 Discrete Fourier Transform

Recalled the exponential form of Fourier series

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{ikw_0 t}$$

$$C_k = \left(\frac{1}{T}\right)\left\{\int_0^T f(t)\times e^{-ikw_0 t}dt\right\}$$

While the above integral can be used to compute C_k, it is more preferable to have a discredited formula version to compute C_k. Further more, the Discrete Fourier Transform (or DFT) [3.1 -3.5] will also facilitate the development of much more efficient algorithms for Fast Fourier Transform (or FFT)

Derivations of DFT Formulas

If time "t" is discretized at $t_1 = \Delta f$, $t_2 = 2\Delta f$, $t_3 = 3\Delta f$,………..$t_n = n\Delta t$

Then Equation (2.18, of Chapter 2) becomes

$$F(t_n) = \sum_{k=0}^{N-1} C_k e^{ikw_0 t_n} \quad (1.10)$$

To simplify the notation, define

$$t_n = n \quad (1.11)$$

Then, Equations (3.11) can be written as

$$f(n) = \sum_{k=0}^{N-1} C_k e^{ikw_0 n} \quad (1.12)$$

In the above formula, "n" is an integer counter. However , f(n) and t_n do NOT have to be integer numbers. Multiplying both sides of Equation (3.12) by $e^{ilw}{}_0n$, and performing the summation of "n", one obtains (note l=integer number)

$$\sum_{n=0}^{N-1} f(n)\times e^{-ilw_0 n} = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} C_k e^{ikw_0 n}\times e^{-ilw_0 n} \quad (1.13)$$

$$= \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} C_k e^{i(k-1)w_0 n} \quad (1.14)$$

$$= \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} C_k e^{i(k-1)\frac{2\pi}{N}n} \quad (1.15)$$

Switching the order of summations on the right hand side of equation (3.15) one obtains

$$\sum_{n=0}^{N-1} f(n)\times e^{-il\left(\frac{2\pi}{N}\right)n} = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} C_k e^{i(k-1)\left(\frac{2\pi}{N}\right)n} \quad (1.16)$$

Define

$$A = \sum_{n=0}^{N-1} e^{i(k-1)\left(\frac{2\pi}{N}\right)n} \quad (`1.17)$$

There are 2 possibilities for (k-1) to be considered in Equation (1.17)

Case (1): (k-1) is a multiple integer of N, such as

(k-1) = mN; or k=1+mN where m = 0, +1, +2……..

Thus, Equation (3.17) becomes:

$$A = \sum_{n=0}^{N-1} e^{im2m}$$

$$= \sum_{n=0}^{N-1} \cos(mn2\pi) + i\sin(mn2\pi) \quad (1.18)$$

Hence: (119)

$$A = N$$

Case (2): (k-1) is NOT a multiple integer of N

In this case, from equation (3.17) one has

$$A = \sum_{n=0}^{N-1}\left\{e^{i(k-1)\left(\frac{2\pi}{N}\right)}\right\}^n \quad (1.20)$$

Define:

$$a = e^{i(k-1)\frac{2\pi}{N}} \quad (1.21)$$

$$= \cos\left\{(k-1)\frac{2\pi}{N})\right\} + i\sin\left\{(k-1)\frac{2\pi}{N})\right\}$$

$a \neq 1$; because (k-1) is "NOT" a multiple integer of N (1.22)

Then, Equation (3.22) can be expressed as

$$A = \sum_{n=0}^{N-1} a^n \quad (1.23)$$

From mathematical handbooks, the right side of Equation (3.23) represents the "geometric series", and can be expressed as

$$A = \sum_{n=0}^{N-1} a^n = N \text{ if } a=1 \quad (1.24)$$

$$= \frac{1-a^N}{1-a} \text{ if } a \neq 1 \quad (1.25)$$

Because of Equation (3.22) , hence Equation (3.25) should be used to compute A, Thus

$$A = \frac{1-a^N}{1-a} \text{(see Equation (3.22)} \quad (1.26)$$

$$= \frac{1-e^{i(k-1)2\pi}}{1-a}$$

Since (k-1) is still a multiple of 2π, hence

$e^{i(k-1)2\pi}=\cos\{(k-1)2\pi\} + i\sin\{(k-1)2\pi\}$ (1.27)

$= 1$

Substituting Equation (3.26) into Equation (3.27), one gets

$A = 0$ (1.28)

Thus, combining the results of case (1) and case (2) , one gets (see equations (3.22) and equation (3.28))

$A = N+0$ (1.29)

Substituting Equation (3.20) in to Equation (3.8), and then referring to Equation (3.7) , one gets

$$\sum_{n=0}^{N-1} f(n)e^{-ilw_0 N} = \sum_{k=0}^{N-1} C_k \times N \quad (1.30a)$$

Recalled k=l+mN(where l, m are integer numbers), and since k must be in the range 0→ N-1, therefore m=0. Thus: k=l+mN becomes k=l

Equation (3.30a) can, therefore, be simplified to

$$\sum_{n=0}^{N-1} f(n)e^{-ilw_0 N} = C_1 \times N \quad (1.30b)$$

Thus

$$C_1 = C_k = \left(\frac{1}{N}\right)\sum_{n=0}^{N-1} f(n)e^{-ilkw_0 n}$$

$$= \left(\frac{1}{N}\right)\sum_{n=0}^{N-1} f(n)\left\{\cos(kw_0 n) - i\sin(kw_0 n)\right\}$$

where

$n \equiv t_n$

and

$$f(n) = \sum_{k=0}^{N-1} C_k e^{ikw_0 n} \quad \text{(3, repeated)}$$

$$= \sum_{k=0}^{N-1} C_k \left\{\cos(kw_0 n) + i\sin(kw_0 n)\right\}$$

Remarks:

a) Consider the exponential term in Equation (3.1) . Let

E= $e^{(ikw_0 n)} = e^{\left(ik\times\frac{2\pi}{N}+n\right)}$

If one replaces “n” by –(N-n)’ (or “(n-N)”) into the above equation , then one obtains

$$e^{ik\times\frac{2\pi}{N}\times(n-N)} = e^{(ik\times\frac{2\pi}{N}+n)} \times\left[e^{(-lk\times 2\pi} = 1\right]$$
$$= \quad E$$

Thus, Equation (3.1) indicates that the force corresponding to frequencies of order “n” and “-(N-n)=n-N” have the same values. Hence

$$w_n = n\overline{W} \text{ for } n \leq \frac{N}{2}$$
$$= -(N-n)\,\overline{W} \text{ for } n > \frac{N}{2}$$

and the frequency corresponding to n= $\frac{N}{2}$ is the highest frequency that can be considered in the discrete Fourier series ($w_{\frac{N}{2}}$ is called the Nyquist frequency). If there are harmonic(force) components above $w_{\frac{N}{2}}$ in the original function, then these higher components will introduce distortions in the lower harmonic components (known as ALIASING phenomenon). Because of the ALIASING phenomenon, the number of (N) data points should be “at least twice” the highest harmonic component presents in the (forcing) function, for sufficient computational accuracy. As an example, if the forcing function is given by

$$F(t) = \sum_{n=1}^{16} 100\times\cos(2\pi nt)$$

then, the minimum value of N(=Number of sample data points) should be N_{mn}= 32.

(b) The factor $\left(\frac{1}{N}\right)$ shown in the DFT equation (3.21) , is merely a scale factor. It can also be placed in the inverse Fourier Transform equation (3.1), but not both.

Thus, Equations (3.21) and (3.1) can be re written as

$$C_n = \sum_{k=0}^{N-1} f(k) e^{-ik\left(w_0=\frac{2\pi}{N}\right)n} \qquad (1.32)$$

$$f(k) = \left(\frac{1}{N}\right)\sum_{k=0}^{N-1} C_n e^{-ik\left(w_0=\frac{2\pi}{N}\right)n} \qquad (1.33)$$

To avoid computation with “complex numbers”, Equation (3.32) can be expressed as

$$C_n^R + iC_n^1 = \sum_{k=0}^{N-1}\left\{f^R(k)+i \quad f'(k)\right\}\times\left\{\cos(\theta)-i\sin(\theta)\right\} \quad (1.32a)$$

Where

$$\theta = k^{\left(w_0=\frac{2\pi}{N}\right)n} \qquad (1.32b)$$

$$C_n^R + iC_n^1 = \sum_{k=0}^{N-1}\left\{f^R(k)\times\cos(\theta)+f'(k)\sin(\theta)\right\}+i\left\{f'(k)\cos(\theta)-f^R(k)\sin(\theta)\right\}$$

The above “complex number” equation is equivalent to the following 2 “real number” equations

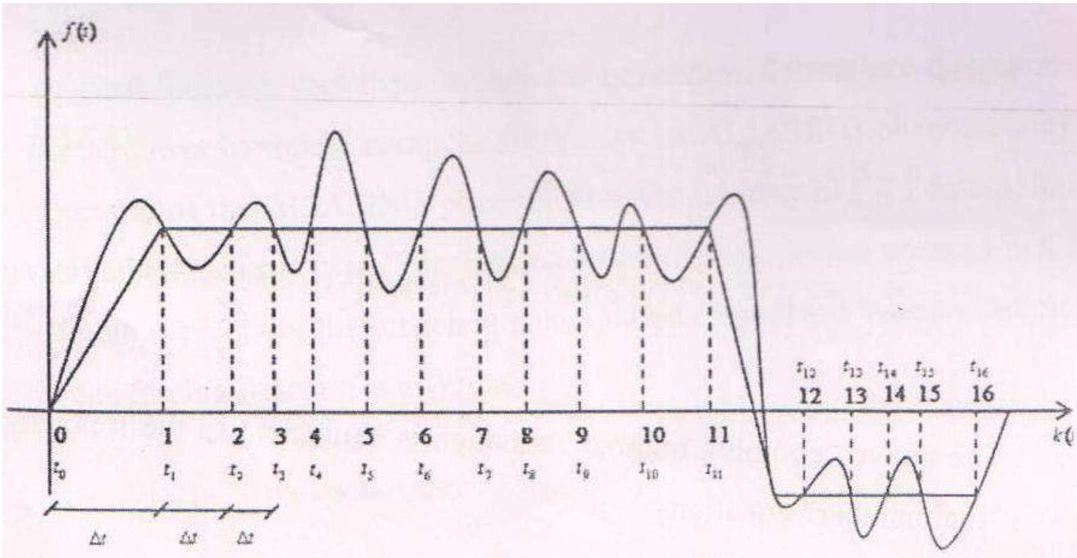
$$C_n^R = \sum_{k=0}^{N-1}\left\{f^R(k)\times\cos(\theta)+f'(k)\sin(\theta)\right\} \qquad (1.32c)$$

$$C_n^1 = \sum_{k=0}^{N-1}\left\{f^1(k)\times\cos(\theta)+f^R(k)\sin(\theta)\right\} \qquad (1.32d)$$

Computer program implementation for the DFT equations (3.32c, 3.32d)

Detailed Explanation about Aliasing Phenomenon, Nyquist Samples, Nyquist Rate.

When a function f(t) which may represent the signals from some real life phenomenon(shown in Figure 1) ,is sampled, it basically converts that function into a sequence f(k) at discrete locations of t. These discrete locations are assumed to have “ equally spaced and the distance between any 2 samples is Δt. Thus f(k) represents the value of f(t), at $t=t_0 + k\Delta t$, where $t_0$is the location of the first sample (at k=0) . IF the sample locations were done properly, then the original function f(t), can be recovered through interpolation process of these discrete sample values.



In Figure 1, the samples have been taken with a fairly large Δt. Thus, these sequence of discrete data will not be able to recover the original signal function f(t). For example, if all discrete values of f(t), were connected by piecewise linear fashion, then a nearly horizontal straight will occur between t_{11} and t_{12} through t_{16}, respectively (see Figure 1). These piecewise linear interpolation (or other interpolation schemes will NOT produce a curve which resemble well with the original function f(t). This is the case where the date has been "ALIASED".

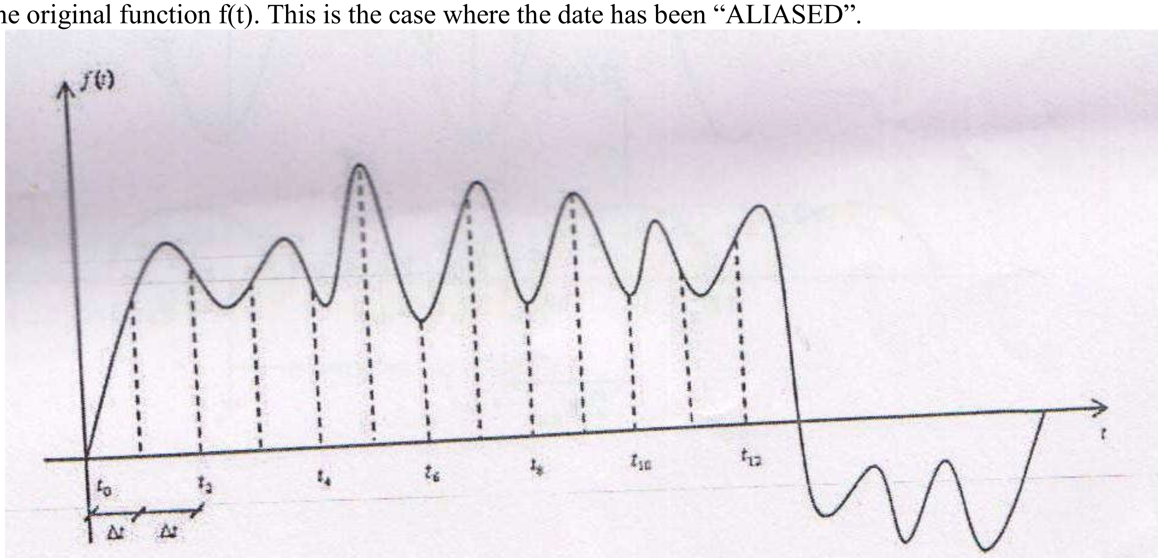


Figure 2 Funciton to be sampled and "Windowing" Sample Problem.

Another potential difficulty in sampling the function is called "windowing" problem. As indicated in Figure 2, while Δt is small enough so that a piecewise linear interpolation for connecting these discrete values will adequately resemble the original function f(t), however, only a portion of the function f(t) has been sampled (from t_1 through t_{12}) rather than the entire one. In other words, one has placed a "widow" over the function.

To avoid aliased phenomenon, the sample space Δt should be small enough so that the discrete sequence will recover back the original function f(t). The "sampling theorem" can be stated as:

" If the function f(t) is band limited with bandwidth $2w_{max}$, F(w)=Fourier transform of f(t)=0 for $|w| \geq w_{max} > 0$ then f(t) is uniquely determined by a knowledge of its values at uniformly spaced intervals Δt apart, with $\Delta t = \frac{1}{2w_{max}}$

The above " sampling theorem" can be loosely explained through the help of Figure 3.

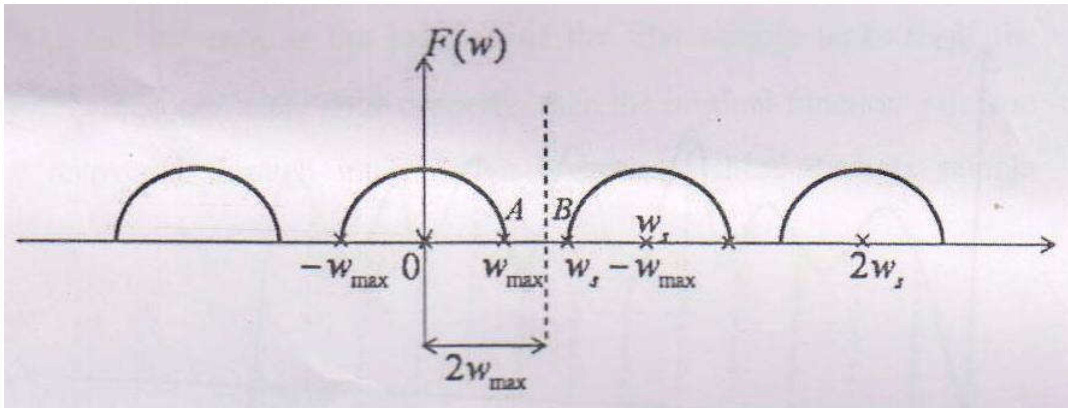


Figure 3 Frequency of sampling rate (w_s) versus maximum frequency content (w_{max}).

To satisfy F(w)=0, for $|w| \geq w_{max}$, the frequency (w) should be between points A and B of Figure 3.

Hence

$$w_{max} \leq w \leq w_s - w_{max}$$

Which implies

$$w_s \geq 2w_{max}$$

Physically, the above equation states that one must have at least 2 samples per cycle of the highest frequency component present (Nyquist samples, Nyquist rate).

Figure 4 Correctly reconstructed signal.

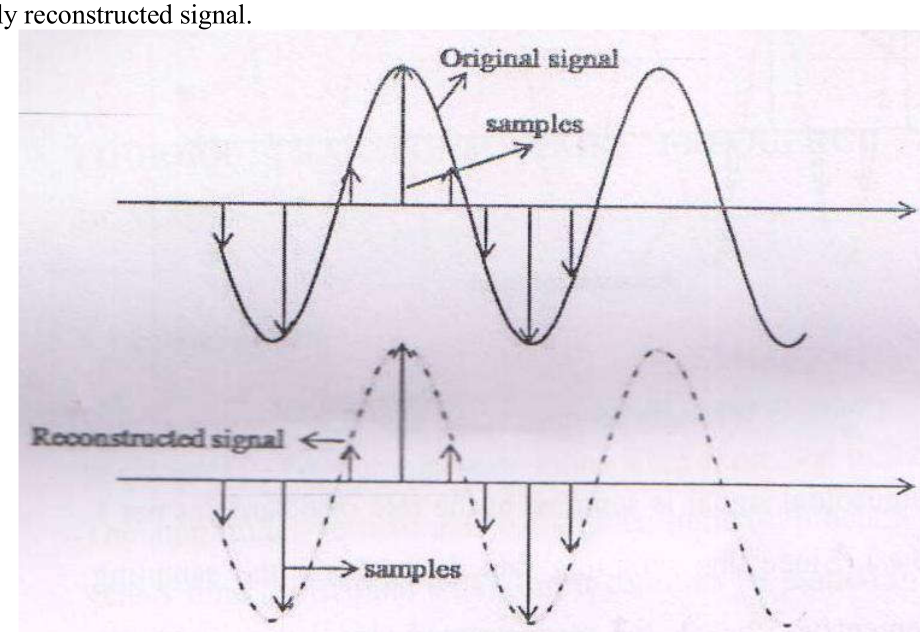
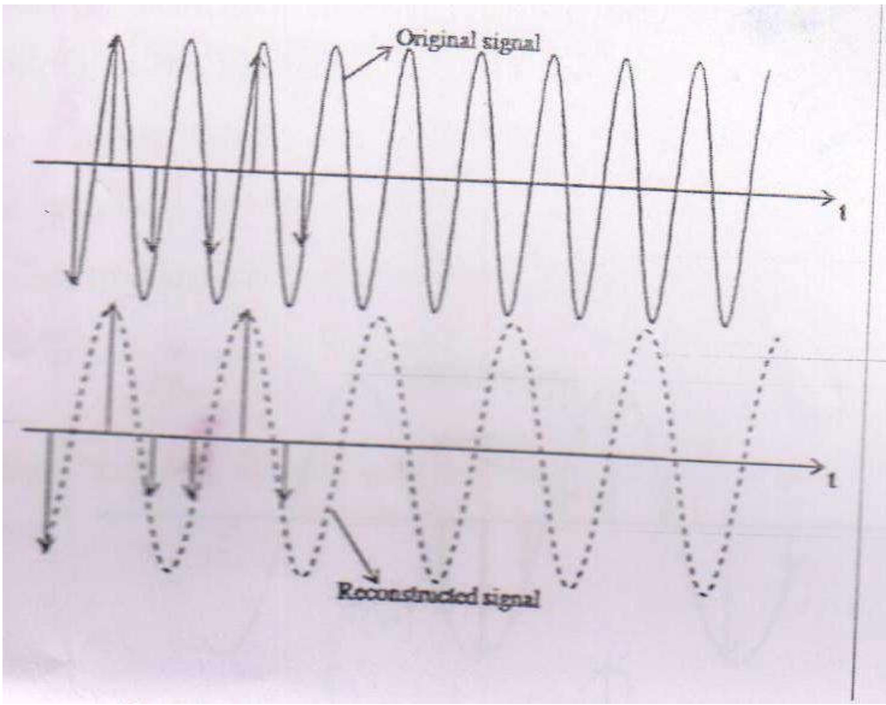


Figure 5 wrongly reconstructed signal.



In Figure 4, a sinusoidal signal is sampled at the rate of 6 samples per 1 cycle (for $w_s = 6w_0$). Since this sampling rate does not satisfy the sampling theorem requirement ($w_s \geq 2w_{\max}$), the reconstructed signal does correctly represent the original signal. However, as indicated in Figure 5 a sinusoidal signal is sampled at the rate of 6 samples per 4 cycles (or $w_s = \frac{6}{4} w_0$). Since this sampling rate does NOT satisfy the requirement ($w_s \geq 2w_{\max}$), the reconstructed signal would wrongly represent the original signal.

REFERENCES:

1. **I. Selesnick and C. S. Burrus, "Automatic generation of prime length FFT programs," IEEE Transactions on Signal Processing, pp. 14–24, Jan. 1996.**
2. **F. Perez and T. Takaoka, "A prime factor FFT algorithm implementation using a program generation technique," IEEE Transactions on Acoustics, Speech and Signal Processing, vol. 35, pp. 1221–1223, August 1987.**
3. **H. W. Johnson and C. S. Burrus, "The design of optimal DFT algorithms using dynamic programming," IEEE Transactions on Acoustics, Speech and Signal Processing, vol. 31, pp. 378–387, Apr. 1983. 16**
4. **Kenneth B. Howell, Principles of Fourier Analysis, (2001), CRC Press.**
5. **R. Bracewell, The Fourier Transform and its Applications, (1999), Tata McGraw-Hill.**
6. **Simon S. Haykin and Van Veen Barry, Signals and Systems, 2nd ed., (2002), Wiley Press.**
7. **Taub and Schilling, Principles of Communication Systems, (1991), Tata McGraw-Hill.**
8. **http://www.math4mobile.com**
9. **http://en.wikipedia.org**