Generalized Semi-TΓ-Ideals in Ternary Γ-Semirings

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Abstract : We introduce the notion of a generalized semi ternary Γ-ideal in aternary Γ-semiring. Various examples to establish a relationship between ternary Γ-ideals, bi ternary Γ-ideals, quasi ternary Γ-ideals and generalized semi ternary Γ-ideals are furnished. Acriterion for a commutative ternary Γ-semiring without any divisors of zeroto a ternary division Γ-semiring is given.

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1. Introduction

Ternary rings and their structures were investigated by Lister [4] in 1971.In fact, Lister characterized those additive subgroups of rings which are closedunder the triple product. In 2003, T. K. Dutta and S. Kar [3] introduced thenotion of a ternary semiring as a generalization of a ternary ring. A ternarysemiring arises naturally as follows. Consider the subset $Z^-$of all negativeintegers of Z. Then $Z^-$is an additive semigroup which is closed under the tripleproduct. $Z^-$is a ternary semiring. Note that Z^-does not form a semiring. In[3] T. K. Dutta and S. Kar introduced the notions of left/right/lateral ideals ofternary semirings and also characterized regular ternary semirings. In 2005, S.Kar [1] introduced the notions of quasi-ideals and bi-ideals in a ternary semiring.The notion of a generalized semi-ideal in a ring has been introduced and studiedby T. K. Dutta in [2].

In 2015 M. Sajani Lavanya and D. Madhusudhana Rao introduced the concepts of ternary Γ-semirings and established the theory of ternary Γ-semirings. In this paper, we introduce the notion of a generalized semiternary Γ-idealin a ternary Γ-semiring and study them. Also, we establish a relationshipbetween generalized semi ternary Γ--ideals, ternary Γ-ideals, bi ternary Γ-ideals, etc. in a ternary Γ-semiring tostudy some properties of a generalized semi ternary Γ-ideals in ternary Γ-semirings.

2. Preliminaries

Definition 2.1: Let T and Γ be two additive commutative semigroups. T is said to be a ***Ternary* Γ*-semiring*** if there exist a mapping from T ×Γ× T ×Γ× T to T which maps $(x_1, \alpha, x_2, \beta, x_3) \to [x_1\alpha x_2\beta x_3]$ satisfying the conditions:

i) $[[a\alpha b\beta c]\gamma d\delta e] = [a\alpha[b\beta c\gamma d]\delta e] = [a\alpha b\beta[c\gamma d\delta e]]$

ii) $[(a + b)\alpha c\beta d] = [a\alpha c\beta d] + [b\alpha c\beta d]$

iii) $[a\alpha (b + c)\beta d] = [a\alpha b\beta d] + [a\alpha c\beta d]$

iv) $[a\alpha b\beta(c + d)] = [a\alpha b\beta c] + [a\alpha b\beta d]$ for all $a, b, c, d \in$ T and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Note 2.2: For the convenience, we write $x_1\alpha x_2\beta x_3$ instead of $[x_1\alpha x_2\beta x_3]$

Note 2.3: Let T be a ternary Γ-semiring. If A,B and C are three subsets of T , we shall denote the set AΓBΓC = $\{\Sigma a\alpha b\beta c : a \in A, b \in B, c \in C, \alpha, \beta \in \Gamma\}$.

Note 2.4: Let T be a ternary Γ-semiring. If A,B are two subsets of T , we shall denote the set A + B = $\{a + b : a \in A, b \in B\}$ and 2A = $\{ a + a : a \in A\}$.

Definition 2.5: An element a of a ternary Γ-semiring T is said to be ***zero*** of T provided $a\alpha b\beta c = b\alpha a\beta c = b\alpha c\beta a = a \ \forall \ b,c \in$ T, $\alpha, \beta \in \Gamma$.

Definition 2.6: A ternary Γ-semiring T is a ***commutative ternary Γ-semiring*** provided $a\alpha b\beta c = b\alpha c\beta a = c\alpha a\beta b = b\alpha a\beta c = c\alpha b\beta a = a\alpha c\beta b$ for all $a,b,c \in$ T and $\alpha, \beta \in \Gamma$.

Definition 2.7: Let T be ternary semiring. A non-empty subset 'S' is said to be a ***ternary* Γ*-subsemiring*** of T if S is an additive sub semigroup of T and $a\alpha b\beta c \in$ S for all $a,b,c \in$ S and $\alpha, \beta \in \Gamma$.

Definition 2.8: An element a of a ternary Γ-semiring T is said to be ***ternary multiplicatively regular*** if there exist $x, y \in$ T and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a\alpha x\beta a\gamma y\delta a = a$.

Definition 2.9: A ternary Γ-semiringTis said to be ***zero divisor free*** (ZDF) if for $a, b, c \in T$, $[a\Gamma b\Gamma c] = 0$ implies that $a = 0$ or $b = 0$ or $c = 0$.

Definition 2.10: Aternary Γ-semiringT is said to be ***division ternary* Γ*-semiring*** if there exists an element b in T such that $a\alpha b\beta x = b\alpha a\beta x = x\alpha a\beta b = x\alpha b\beta a = x$ for all $x \in$T and $\alpha, \beta \in \Gamma$.

Definition 2.11: A nonempty subset A of a ternary Γ-semiring T is said to be ***ternary* Γ*-ideal*** or simply *T**Γ**-ideal* of T if (1) $a, b \in$ A implies $a + b \in$ A, (2) $b, c \in$ T,, $\beta \in \Gamma$, $a \in$ A⇒$b\alpha c\beta a \in$ A, $b\alpha a\beta c \in$ A, $a\alpha b\beta c \in$ A.

Definition 2.12: An additivesub-semigroup Q ofa ternaryΓ-semiring T is calleda ***quasi-ternary Γ-ideal*** of T if QΓTΓT∩(TΓQΓT+TΓTΓQΓTΓT)∩TΓTΓQ⊆Q.

Definition 2.13: A ternary Γ-subsemiring B of a ternary Γ-semiring T is called a ***bi-ternary* Γ*-ideal*** of T if BΓTΓBΓTB⊆B.

3. Generalized semi TΓ-ideals in ternary Γ-semirings

Generalized semi-ideals in ternary semirings are introduced and studied by daddi, V. R, Pawar. V. S in [2]. As a generalization, we define generalized semi TΓ-ideals in ternaryΓ-semirings.

Definition 3.1. A non-empty subset A of Tsatisfying the condition $a + b \in A$,for all $a; b \in A$ is called

i) generalized left semi ternary Γ-ideal of Tif $[[x\Gamma x\Gamma x]\Gamma x\Gamma a] \subseteq A$ for all $a \in A, x \in T$,

ii) generalized lateral semi ternary Γ-ideal of T if $[x\Gamma[x\Gamma a\Gamma x]\Gamma x] \subseteq$ A for all $a\in$ A, $x\in$ T,

iii) generalized right semi ternary Γ-ideal of Tif $[a\Gamma x\Gamma x]\Gamma x\Gamma x] \subseteq A$ for all $a \in A$,$x \in T$.

iv) generalized semi ternary Γ-ideal of T if A is generalized left semi ternary Γ-ideal, generalized lateral semi ternary Γ-ideal, generalized right semi ternary Γ-ideal of T.

Example 3.2: Let T= {...,−2i, −i, 0, i, 2i, ...} and Γ = Tbe a ternary Γ-semiring with respect to addition and complex triple multiplication.Let A = {0, i, 2i, ...}. $A$is a generalized semi ternary Γ-ideal of T.

Note 3.3: The concepts of generalized semi ternary Γ-ideal and ternary Γ-subsemiringare independent in T. i.e. every ternary Γ-subsemiring of Tneednot be a generalized semi ternary Γ-ideal of $T$and every generalized semi ternary Γ-ideal of Tneednot be a ternary Γ-subsemiring of T. For this, consider the following examples.

Example 3.4: Let $T= M_2(Z_0^-)$ and Γ = T, then T be the ternary Γ-semiring of the set of all 2x2 square matrices over Z_0^-, the set of all non-positive integers.

Let $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} / a \in Z_0^- \right\}$. Then $S$is a ternary Γ-subsemiring of T. But this is not a generalized semi ternary Γ-ideal of T.

Example 3.5: In example 3.2, A is generalized semi ternary Γ-ideal of T. But it is not a ternary Γ-subsemiring of T.

Remark 3.6: The intersection of an arbitrary collection of generalized semi ternary Γ-ideals of aternary Γ-semiring is generalized semi ternary Γ-ideal of T. But, the union of two generalizedsemi ternary Γ-ideals of Tmay not be a generalized semi ternary Γ-ideal of T. This we establish inthe following example.

Example 3.7: Let T= {. . . .−2i,−i,0, I,2i, . . .} and T = Γbe a ternary Γ-semiring with respect toaddition and complex triple multiplication. Then I = {....−4i,−2i,0,2i, ...}and J = {……−10i,−5i,0,5i,10i, ...}are two generalized semi ternary Γ-ideals of T, but$I\cup J$ is not a generalized semi ternary Γ-ideal of T.

Theorem 3.8:Let A be a generalized semi ternary Γ-ideal of T and let S be a ternary Γ-subsemiring of T. If A∩T ≠∅, then A∩T is a generalized semi ternary Γ-ideal of T.

Proof. Let a, $b \in A\cap T$. Then $a+b\in A\cap T$. For $x \in T$ and $a \in A\cap T$ we have$[[x\Gamma x\Gamma x]\Gamma x\Gamma a] \subseteq A\cap T$, $[[a\Gamma x\Gamma x]\Gamma x\Gamma x] \subseteq A\cap T$,$[[x\Gamma x\Gamma a]\Gamma x\Gamma x] \subseteq A\cap T$.Hence $A\cap T$ is generalized semi ternary Γ-ideal of T.

Theorem 3.9: If A and B are generalized semi ternary Γ-ideals of T, then A + B = {a + b / a ∈ A; b ∈ B} is a generalized semi ternary Γ-ideal of T.

Proof: Let $x, y \in A + B$. Hence $x = a + b$, $y = c + d$, for $a, c \in A$ and $b, d \in B$.

Then $x + y = (a + b) + (c + d) = (a + c) + (b + d) \in A + B$. Let $t \in$ T, $\alpha, \beta, \gamma, \delta \in \Gamma$ and

$x \in A + B$, hence $x = a + b$ for some $a \in A$ and $b \in B$. Therefore

$[[t\alpha t\beta t]\gamma t\delta x] = [[t\alpha t\beta t]t\delta(a + b)] = [[t\alpha t\beta t]\gamma(t\delta a)] + [[t\alpha t\beta t]\gamma(t\delta b)] \in A + B$.

Similarly, we have

$[[t\alpha t\beta x]t\delta t] = [[t\alpha t(a + b)]\gamma t\delta t] = [([t\alpha t\beta a] + [t\alpha t\beta b])\gamma t\delta t]$

$= [[t\alpha t\beta a]t\delta t] + [[t\alpha t\beta b]\gamma t\delta t] \in A + B$ and

$[[x\alpha t\beta t]t\delta t] = [[(a + b)t\beta t]$□$t$□$t] = [([a$□$t$□$t] + [b$□$t$□$t])$□$t$□$t]$

$= [[a$□t□$t]\gamma t$□$t] + [[b$□t□$t]$□t□$t] \in A + B$. Thus $A + B$ is a generalized semi ternary □-ideal of T.

Theorem 3.10: Let T be a ternary □-semiring with zero. Let A and B be two generalized semi ternary □-ideals of T containing zero. Then A+B is the smallest generalized semi ternary □-ideal of T containing both A and B.

***Proof*:** From Theorem 3.9, $A + B$ is a generalized semi-ternary Γ-ideal of T. Since $0 \in A$, $0 \in B$ we get $0 \in A + B$ and for $a \in A$, $a = a + 0 \in A + B$. Hence $A \subseteq A + B$.

Similary, $B \subseteq A+B$. Let I be any other generalized semi ternary Γ-ideal containing both A and B. Let $x \in A + B$. Then $x = a + b$, for some $a \in A$ and $b \in B$. Hence, $x = a + b \in I$. Therefore, $A + B \subseteq I$. Thus, $A + B$ is the smallest generalized semi ternary Γ-ideal containing both A and B.

Theorem 3.11: Let A be a generalized left semi ternary □-ideal of T. Then [A□B□C] is a generalized left semi ternary □-ideal for any non-empty subsets B and C of T.

***Proof*:** For x, y □ $[A\Gamma B\Gamma C]$, let $x = \sum_{i=1}^{n} a_i\alpha b_i\beta c_i$ and $y = \sum_{j=1}^{m} a_j\alpha b_j\beta c_j$. Obviously $x + y$ is a finite sum of the form $\sum_{k=1}^{p} a_k\alpha b_k\beta c_k$. Hence $x + y$ □ $[A\Gamma B\Gamma C]$. For t □ T, we have

$$[[t\Gamma t\Gamma t]\Gamma t\Gamma x] = [[t\Gamma t\Gamma t]\Gamma t\Gamma \sum_{i=1}^{n} a_i\alpha b_i\beta c_i\ ] = \sum_{i=1}^{n} t\Gamma t\Gamma t\Gamma t\Gamma(a_i\alpha b_i\beta c_i)$$

$= \sum_{i=1}^{n}[[[t\Gamma t\Gamma t]\Gamma t\Gamma a_i]\alpha b_i\beta c_i]$ □ $[A\Gamma B\Gamma C]$. Since A is generalized left semi ternary Γ-ideal T, $[A\Gamma B\Gamma C]$ is a generalized left semi ternary Γ-ideal of T.

Theorem 3.12: Let A and B be ternary □-subsemirings of T such that A□A□A = A and A be a left ternary □-ideal of B and B be a generalized left semi ternary □-ideal of T. Then A is a generalized left semi ternary □-ideal of T.

Proof. Let a □ A, therefore $a = [a_1$□a_2□$a_3]$, where a_1, a_2, a_3 □ A, □, □ □ Γ. Now, for any x □ T, □, □, □, □ □ Γ, $[x$□x□$x]\gamma x$□$a] = [[x$□x□$x]$□x□$[a_1$□a_2□$a_3]]$

$= [[[x$□x□$x]x$□$a_1]$□a_2□$a_3]$ □ $[B\Gamma a_2\Gamma a_3]$ □ A.

(since A is a left ternary Γ-ideal of B, a_1 □ A □ B, B is a generalized left semi ternary Γ-ideal of T). Therefore, A is a generalized left semi ternary Γ-ideal of T.

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