ARCHIMEDIAN Γ-SEMIGROUPS

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Abstract: In this paper the terms the terms; Archimedean Γ-semigroup and strongly Archimedean Γ-semigroup are introduced. It is proved that if S is a duo **Γ**-semigroup, then the conditions (1) S is strongly Archimedean, (2) S is Archimedean, (3) S has no proper completely prime **Γ**-ideals and (4) S has no proper prime **Γ**-ideals; are equivalent. In an archimedian duo Γ-semigroup S, if S is a union of finite number of principal Γ-ideals or S contains a maximal Γ-ideal which is finitely generated, then it is proved that every proper Γ-ideal is principal and S is a union of at most two principal Γ-ideals.

Keywords: **Duo Γ-semigroup, Archimedean Γ-semigroup, strongly Archimedean Γ-semigroup**

PRELIMINARIES

DEFINITION 2.1 : Let S and Γ be any two non-empty sets. Then S is said to be a **Γ-*semigroup*** if there exist a mapping from $S\times\Gamma\times S$ to S which maps $(a, \gamma, b) \to a\ \gamma\ b$ satisfying the condition : $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

NOTE 2.2 : Let S be a Γ-semigroup. If A and B are two subsets of S, we shall denote the set $\{ a\gamma b : a \in A , b \in B \text{ and } \gamma \in \Gamma \}$ by AΓB.

DEFINITION 2.3 : A nonempty subset A of a Γ-semigroup S is said to be a ***left* Γ-*ideal*** of S if $s \in S, a \in A, \alpha \in \Gamma$ implies $s\alpha a \in A$.

NOTE 2.4 : A nonempty subset A of a Γ-semigroup S is a left Γ- ideal of S iff SΓA⊆A.

DEFINITION 2.5 : A nonempty subset A of a Γ-semigroup S is said to be a ***right* Γ-*ideal*** of S if $s \in S, a \in A, \alpha \in \Gamma$ implies $a\alpha s \in A$.

NOTE 2.6 : A nonempty subset A of a Γ-semigroup S is a right Γ- ideal of S iff AΓS⊆A.

DEFINITION 2.7 : A nonempty subset A of a Γ-semigroup S is said to be a ***two sided* Γ- *ideal*** or simply a **Γ- *ideal*** of S if s ∈ S, a ∈ A, α ∈ Γ imply $s\alpha a$ ∈ A, $a\alpha s$ ∈ A.

NOTE 2.8 : A nonempty subset A of a Γ-semigroup S is a two sided Γ-ideal iff it is both a left Γ-ideal and a right Γ- ideal of S.

THEOREM 2.9 : The nonempty intersection of any two (left or right) Γ-ideals of a Γ-semigroup S is a (left or right) Γ-ideal of S.

THEOREM 2.10 : The nonempty intersection of any family of (left or right) Γ-ideals of a Γ-semigroup S is a (left or right) Γ-ideal of S.

THEOREM 2.11 : The union of any two (left or right) Γ-ideals of a Γ-semigroup S is a (left or right) Γ-ideal of S.

THEOREM 2.12 : The union of any family of (left or right) Γ-ideals of a Γ-semigroup S is a(left or right) Γ-ideal of S.

DEFINITION 2.13 : A Γ- semigroup S is said to be a ***left duo Γ- semigroup*** provided every left Γ- ideal of S is a two sided Γ- ideal of S.

DEFINITION 2.14 : A Γ- semigroup S is said to be a ***right duo* Γ- *semigroup*** provided every right Γ- ideal of S is a two sided Γ- ideal of S.

DEFINITION 2.15 : A Γ- semigroup S is said to be a ***duo* Γ- *semigroup*** provided it is both a left duo Γ- Semigroup and a right duo Γ- semigroup.

THEOREM 2.16 : A Γ- semigroup S is a duo Γ- semigroup if and only if $x\Gamma S^1 = S^1\Gamma x$ for all $x \in S$.

THEOREM 2.17 : Let A be a Γ-ideal in a duo Γ-semigroup S and $a, b \in S$. Then $a\Gamma b \subseteq A$ if and only if $< a > \Gamma < b > \subseteq A$.

COROLLARY 2.18 : Let A be a Γ-ideal in a duo Γ-semigroup S. Then for any natural number n, $(a\Gamma)^{n-1}a \subseteq A$ implies $(< a > \Gamma)^{n-1} < a > \subseteq A$.

DEFINITION 2.19 : A Γ-ideal A of a Γ-semigroup S is said to be a ***maximal* Γ*-ideal*** provided A is a proper Γ-ideal of S and A is not properly contained in any proper Γ-ideal of S.

DEFINITION 2.20 : A Γ- ideal P of a Γ-semigroup S is said to be a ***completely prime* Γ*- ideal*** provided $x, y \in S$ and $x\Gamma y \subseteq P$ implies either $x \in P$ or $y \in P$.

DEFINITION 2.21 : A Γ- ideal P of a Γ-semigroup S is said to be a ***prime* Γ*- ideal*** provided A, B are two Γ-ideals of S and $A\Gamma B \subseteq P \Rightarrow$ either $A \subseteq P$ or $B \subseteq P$.

COROLLARY 2.22 : A Γ- ideal P of a Γ-semigroup S is a prime Γ- ideal iff $a, b \in S$ such that $a\Gamma S^1\Gamma b \subseteq P$, then either $a \in P$ or $b \in P$.

THEOREM 2.23 : Let S be a duo Γ-semigroup. A Γ-ideal P of S is prime Γ-ideal if and only if P is a completely prime Γ-ideal.

DEFINITION 2.24 : If A is a Γ-ideal of a Γ-semigroup S, then the intersection of all prime Γ-ideals of S containing A is called ***prime* Γ*-radical*** or simply **Γ*-radical*** of A and it is denoted by $\sqrt{A}$ or ***rad A***.

DEFINITION 2.25 : If A is a Γ-ideal of a Γ-semigroup S, then the intersection of all completely prime Γ-ideals of S containing A is called ***complete prime* Γ*-radical*** or simply ***complete* Γ*-radical*** of A and it is denoted by ***c. rad A***.

NOTE 2.26 : If A is a Γ-ideal of a Γ-semigroup S then ***rad*** $A = A_3$ and ***c.rad*** $A = A_4$.

THEOREM 2.27 : If A is a Γ-ideal of a duo Γ-semigroup S, then *rad* A = *c.rad* A.

NOTATION 2.28 : If A is a Γ-ideal of a Γ-semigroup S, then we associate the following four types of sets.

A_1 = The intersection of all completely prime Γ-ideals of S containing A.

$A_2 = \{x \in S : (x\Gamma)^{n-1}x \subseteq A$ for some natural number $n\}$

A_3 = The intersection of all prime ideals of S containing A.

$A_4 = \{x \in S : (<x>\Gamma)^{n-1}<x> \subseteq A$ for some natural number $n\}$

COROLLARY 2.29 : If A is a Γ-ideal in a duo Γ-semigroup S and $x, y \in S$, then $x\Gamma y \subseteq A$ implies $x\Gamma s\Gamma y \subseteq A$.

THEOREM 2.30 : If A is a Γ-ideal in a duo Γ-semigroup S, then $A_l(a) = \{x \in S : x\Gamma a \subseteq A\}$ and $A_r(a) = \{x \in S : a\Gamma x \subseteq A\}$ are Γ-ideals of S for all $a \in S$.

THEOREM 2.31 : Let A be a Γ-ideal in a duo Γ-semigroup S and a, b ∈ S. Then $a\Gamma b \in A$ if and only if $< a > \Gamma < b > \subseteq A$.

THEOREM 2.32 : Let A be a Γ-ideal in a duo Γ-semigroup S. Then $a_1\Gamma a_2\Gamma \ldots a_{n-1}\Gamma a_n \subseteq A$ if and only if $< a_1 > \Gamma < a_2 > \ldots \Gamma < a_n > \subseteq A$.

COROLLARY 2.33 : Let A be a Γ-ideal in a duo Γ-semigroup S. Then for any natural number n, $(a\ \Gamma)^{n-1}a \subseteq A$ if and only if $(< a > \Gamma)^{n-1} < a > \subseteq A$.

THEOREM 2.34 : Let S be a duo Γ-semigroup. A Γ-ideal P of S is prime Γ-ideal if and only if P is a completely prime Γ-ideal.

COROLLARY 2.35 : If H is the collection of all Γ-ideals in a Γ-closed duo Γ-semigroup S, which are not finitely generated and H≠∅, then there exists a prime Γ-ideal which is not finitely generated.

3. ARCHIMEDIAN Γ-SEMIGROUPS

DEFINITION 3.1 : A Γ- semigroup S is said to be an ***archimedian* Γ*- semigroup*** provided for any $a,b \in S$, there exists a natural number n such that $(a\Gamma)^{n-1}a \subseteq < b >$.

DEFINITION 3.2 : A Γ-semigroup S is said to be a ***strongly archimedean*** **Γ*-semigroup*** provided for any a, b ∈ S, there is a natural number n such that $(< a >\Gamma)^{n-1} < a > \subseteq\ < b >$.

THEOREM 3.3 : If S is a duo Γ-semigroup, then S is strongly archimedean if and only if archimedean.

Proof : Suppose that S is strongly Archimedean.

Then for any $a,b \in S$, there is a natural number n such that $(< a >\Gamma)^{n-1} < a > \subseteq\ < b >$.

Therefore $(a\Gamma)^{n-1}a \subseteq (< a >\Gamma)^{n-1} < a > \subseteq < b >$ and hence S is Archimedean. Conversely suppose that S is archemedian. Let $a,b \in S$. Since S is archemedian, there exists a natural number n such that

$(< a >\Gamma)^{n-1} < a > \subseteq < b > \subseteq S\,\Gamma b\,\Gamma S$. Since $S\,\Gamma b\,\Gamma S$ is a Γ-ideal of a duo Γ-semigroup S, by corollary 3.2.5, $(a\Gamma)^{n-1}a \subseteq S\,\Gamma b\,\Gamma S$

$\Rightarrow (< a > \Gamma)^{n-1} < a > \subseteq S\,\Gamma b\,\Gamma S$. Therefore S is a strongly Archimedean duo **Γ**-semigroup**.**

THEOREM 3.4 : If S is a duo Γ-semigroup, then S is archimedean if and only if S has no proper prime Γ-ideals.

Proof : Suppose that S is archimedean Γ-semigroup. Let *P* be prime Γ-ideal of *S*. Let *a,b* ∈ S. Since *P* is Γ-ideal, $S\Gamma a\Gamma S \subseteq P$. Since *S* is archemedian, $(b\Gamma)^{n-1} \subseteq S\Gamma a\Gamma S$ for some natural number *n*. Thus $(b\Gamma)^{n-1} \subseteq S\Gamma a\Gamma S \subseteq P$. Since *S* is a duo Γ-semigroup, by theorem 3.2.10, *P* is completely prime. Thus $(b\Gamma)^{n-1}b \subseteq P \Longrightarrow$ *b* ∈ P. Hence *S* =*P*. Therefore *S* has no proper prime Γ-ideals.

Conversely suppose that S has no proper prime Γ-ideals. Then for any *b* ∈ S, the intersection of all prime Γ-ideals of *S* containing *B* = < *b* > is *S* itself. Therefore $B_3 = S$. We have $B_4 = \{x \in S : (< x > \Gamma)^{n-1} < x > \subseteq < b >$ for some $n \in N\}$

= *S*.

Therefore for any *a* ∈ S, $(< a >\Gamma)^{n-1} < a > \subseteq < b >$ for some natural number *n*.

So $(< a >\Gamma)^{n-1} < a > \subseteq S\Gamma b\Gamma S$. Thus S is strongly archimedian.

Hence by theorem 3.3, S is archimedian.

COROLLARY 3.5 : If S is a duo Γ-semigroup, then the conditions (1) S is strongly Archimedean, (2) S is Archimedean, (3) S has no proper completely prime Γ-ideals and (4) S has no proper prime Γ-ideals are equivalent.

THEOREM 3.6 : Let S be a duo archemedian Γ-semigroup. If S is a union of finite number of principal Γ-ideals, then every proper Γ-ideal is principal and S is a union of at most two principal Γ-ideals.

Proof **:** Suppose that S $= \bigcup_{i=1}^{n} < x_i >$. Let H be the collection of all proper Γ-ideals which are not principal. If H $\neq \emptyset$ then clearly H is a partially ordered set under set inclusion. Let $\{A_\alpha\}$ be a chain of Γ-ideals in H. If S = $\bigcup A_\alpha$ then $x_i \in A_i$ for some natural number *i*.

If we take *j* = max $\{1, 2, 3, \ldots, n\}$ then $x_i \in A_j$ for *i* = 1, 2, 3, …, *n*. So S = $\bigcup_{i=1}^{n} < x_i > \subseteq A_j \subseteq S$. and hence A_j= S.

It is a contradiction. Hence S $\neq \bigcup A_\alpha$. If $\bigcup A_\alpha = < a >$ for some $a \in S$, then $a \in A_i$ for some *i* and hence $A_i = < a >$, which is not true. Thus $\bigcup A_\alpha \in$ H. Therefore H satisfies the hypothesis of Zorn`s lemma. By Zorn`s lemma, there exists a maximal element *P* in H. By corollary 3.8, P is a prime Γ-ideal of S. Since S is a duo archemedian Γ-semigroup, by theorem 3.4, S has no proper prime Γ-ideals. It is a contradiction. Hence H = $\emptyset$. Therefore every proper Γ-ideal of S is a principal Γ-ideal. Let S = $\bigcup_{i=1}^{n} < x_i >$ with $x_i \notin < x_j >$ for $i \neq j$. If $n > 2$, then $S \neq < x_1 > \cup < x_2 >$. Since $< x_1 > \cup < x_2 >$;is a proper Γ-ideal, $< x_1 > \cup < x_2 >$ is a principal Γ-ideal. Thus either $< x_1 > \subseteq < x_2 >$ or $< x_2 > \subseteq < x_1 >$. This contradicts the choice of x_i`s. Thus $n \leq 2$,

THEOREM 3.7 : Let S be an archemedian duo Γ-semigroup. If S contains a maximal Γ-ideal which is finitely generated, then every proper Γ-ideal is principal and S is a union of at most two principal Γ-ideals.

Proof **:** Suppose that S contains a maximal Γ-ideal M which is finitely generated. Let $a \in S/M$. Since M is maximal, $S = M \cup < a >$. So S is a union of finite number of principal Γ-ideals. Therefore by theorem 3.6, every Γ-ideal is principal and S is a union of at most two principal Γ-ideals.

DEFINITION 3.7 : A Γ-semigroup S is said to be a ***noetherian Γ-semigroup*** if ascending chain of Γ-ideals becomes stationary. i.e., if $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$is an ascending chain of Γ-ideals of S, then there exists a natural number *m* such that $A_m = A_n$ for all natural numbers $n \geq m$.

NOTE 3.8 : A Γ-semigroup S is noetherian if and only if every Γ-ideal of S is a union of finite number of principal Γ-ideals of S.

THEOREM 3.9 : If S is a noetherian Γ-semigroup containing proper Γ-ideals then S has a maximal Γ-ideal.

Proof **:** Let A_1 be a proper Γ-ideal of S. If A_1 is not a maximal Γ-ideal of S, then there exists a proper Γ-ideal A_2 of S such that $A_1 \subset A_2$. If A_2 is not a maximal Γ-ideal of S, then there exists a proper Γ-ideal A_3 of S such that $A_1 \subset A_2 \subset A_3$. By continuing this process we get an ascending chain of proper Γ-ideals of S. Since S is noetherian, the chain $A_1 \subset A_2 \subset A_3 \ldots$ is stationary. It is a contradiction. Therefore there exists a maximal Γ-ideal of S.

THEOREM 3.10 : Let S be a duo noetherian Γ-semigroup such that S = $\bigcup_{i=1}^{n} < x_i >$. Suppose $a \notin < x_i \Gamma a >$ for all $a \in S$, which is not a product of power of x_i`s. Then S is finitely generated. In particular if S is noetherian strongly Γ-cancellative Γ-semigroup without identity then S is finitely generated.

Proof **:** Suppose that there exists an element *a* such that *a* is not a product of x_i`s. If *a* = $x_i \alpha_1 s_1$ for $\alpha_1 \in \Gamma$, where $a \neq s_1$ is not a product of power of $x_i$`s. Hence $s_1 = x_j \alpha_2 s_2$ for $\alpha_2 \in \Gamma$, where s_2 is not product of powers of x_i`s. If $s_2 \in < s_1 >$ then $s_2 = s_1 \alpha_3 r$ for some $r \in S^1$, $\alpha_3 \in \Gamma$ and hence $s_1 = x_j \alpha_2 (s_1 \alpha_3 r) \in < x_j \Gamma s_1 >$, which is not true. Hence $< s_1 > \subset < s_2 >$. By continuing this process, we get a nonterminating chain of Γ-ideals

$< s_1 > \subset < s_2 > \subset < s_3 > \subset \cdots$ Since S is noetherian, it is a contradiction. So S is finitely generated. If S is a strongly Γ-cancellative Γ-semigroup and if $a = a\beta_1(b\beta_2 a)$ for $\beta_1, \beta_2 \in \Gamma$, then $b\beta_2 a$ is an identity in S. It is a contradiction. So $a \notin < x_i \Gamma a >$ for all $a \in S$. As above, we have S is finitely generated.

THEOREM 3.11 : Let S be an archemedian duo Γ-semigroup with S $= \bigcup_{i=1}^{n} < x_i >$. If $a \notin < x_i \Gamma a >$ for all $a \in S$, which is not a product of powers of x_i`s, then S is finitely generated.

Proof **:** Let S be an archemedian duo Γ- semigroup with S $= \bigcup_{i=1}^{n} < x_i >$. By theorem 3.6, S is a union of at most two principal Γ-ideals. By theorem 3.10, S is finitely generated.

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