Legendre Wavelet Collocation Method for the Numerical Solution of Integral and Integro-Differential Equations

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Abstract:

Legendre wavelet collocation method for the numerical solution of Fredholm, Volterra, mixed Volterra-Fredholm integral equations, integro-differential equations and weakly singular Fredholm integral equations. The present scheme is based upon Legendre polynomials and Legendre wavelet approximations. The properties of Legendre wavelet is first presented and the resulting Legendre wavelet matrices are utilized to reduce the integral and integro-differential equations into system of algebraic equations, then the required Legendre coefficients are computed using Matlab. Some of the numerical examples are tested and compared with exact and existing methods. Error analysis is worked out, which shows efficiency of the proposed method.

Keywords: Legendre wavelet, Collocation method, Integral equations, Integro-differential equations.

1. INTRODUCTION

Integral and integro-differential equation has several applications in various fields of science and engineering. There are different numerical methods for approximating the solution of integral and integro-differential equations are known [1]. Wavelets theory is a moderately new and an emerging tool in applied mathematical research area. It has been applied in a wide range of engineering disciplines; particularly, signal analysis for waveform representation and segmentations, time-frequency analysis and fast algorithms for easy implementation. Wavelets allow the precise representation of a variety of functions and operators. Moreover, wavelets establish a connection with fast numerical algorithms [2, 3]. Since 1991 the various types of wavelet method have been applied for the numerical solution of different kinds of integral equations, a detailed survey on these papers can be found in [4]. For solving these equations, such as Lepik et al. [5-11] applied the Haar wavelet method. Maleknejad et al. [12-16] has introduced rationalized haar wavelet, Legendre wavelet, Hermite Cubic spline wavelet, and Coifman wavelet. Babolian and Fattahzadeh [17] have applied chebyshev wavelet operational matrix of integration. Abdalrehman [18] has proposed an algorithm for n^{th} order integro-differential equations by using Hermite Wavelets Functions. Yousefi and Banifatemi [19] has proposed a recently CAS wavelet. Ramane et al. [27] have applied a new Hosoya polynomial of path graphs for the numerical solution of Fredholm integral equations. In this paper, we proposed the Legendre wavelet (LW) collocation method for the numerical solution of integral and integro-differential equation. The proposed method is explained and demonstrated the efficiency of the scheme than the existing method by presenting some of the illustrative examples.

2. Properties of Legendre wavelets

2.1 Wavelets

In recent years, wavelets have found their way into many different fields of science and engineering. Wavelets constitute a family of single functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets [20, 21].

$$\psi_{a,b}(t)=|a|^{-\frac{1}{2}}\psi\left(\frac{t-b}{a}\right),\quad a,b\in R,\quad a\neq 0$$

2.2 Legendre wavelets

Legendre wavelet $L_{n,m}(t)=L(k,\hat{n},m,t)$ have four arguments; $k=2,3,\ldots,\ \hat{n}=2n-1,$

$n=1,2,3,\ldots,2^{k-1},m$ is the order of the Legendre polynomials and t is the normalized time. They are defined on the interval [0,1) by:

$$L_{n,m}(t)=\begin{cases}(m+\frac{1}{2})^{\frac{1}{2}}2^{k/2}l_m(2^k t-\hat{n}), & \frac{\hat{n}-1}{2^k}\leq t<\frac{\hat{n}+1}{2^k},\\ 0, & otherwise.\end{cases} \tag{2.1}$$

Here, $l_m(t)$ are the well-known Legendre polynomial of order m, which are orthogonal with respect to the weight function $w(t)=1$ and satisfy the following recursive formula:

$$\begin{aligned}&l_0(t)=1,\\&l_1(t)=t,\\&l_{m+1}(t)=\frac{2m+1}{m+1}\,t\,l_m(t)-\frac{m}{m+1}l_{m-1}(t),\quad m=1,2,3,\ldots\end{aligned}$$

The set of Legendre wavelets are an orthonormal set (Razzaghi (2000, 2001)).

The six basis functions are given by:

$$\left.\begin{aligned} L_{10}(t) &= \sqrt{2} \\ L_{11}(t) &= \sqrt{6}\,(4t-1) \\ L_{12}(t) &= \sqrt{10}\left(\frac{3}{2}(4t-1)^2 - \frac{1}{2}\right) \end{aligned}\right\}; \; 0 \le t < \frac{1}{2},$$

$$\left.\begin{aligned} L_{20}(t) &= \sqrt{2} \\ L_{21}(t) &= \sqrt{6}\,(4t-3) \\ L_{22}(t) &= \sqrt{10}\left(\frac{3}{2}(4t-3)^2 - \frac{1}{2}\right) \end{aligned}\right\}; \; \frac{1}{2} \le t < 1$$

For $k = 2$ implies $n = 1, 2$ and M = 3 implies $m = 0, 1, 2$ then using collocation points $t_j = \frac{j-0.5}{N}, j = 1, 2, \ldots, N$, Eq.(2.1) gives the Legendre wavelet matrix of order $(N = 2^{k-1}M)$ 6x6 as,

$$L_{6\times 6} = \begin{bmatrix} 1.4142 & 1.4142 & 1.4142 & 0 & 0 & 0 \\ -1.6330 & 0 & 1.6330 & 0 & 0 & 0 \\ 0.5270 & -1.5811 & 0.5270 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.4142 & 1.4142 & 1.4142 \\ 0 & 0 & 0 & -1.6330 & 0 & 1.6330 \\ 0 & 0 & 0 & 0.5270 & -1.5811 & 0.5270 \end{bmatrix}$$

3. Legendre Wavelet Collocation Method of Solution

In this section, we present a Legendre wavelet (LW) collocation method for solving integral and integro-differential equation,

3.1 Integral Equations

Fredholm Integral equations:

Consider the Fredholm integral equations,

$$u(t) = f(t) + \int_0^1 k_1(t,s)\,u(s)\,ds, \tag{3.1}$$

where $f(t) \in L^2[0,1)$, $k_1(t,s) \in L^2([0,1)\times[0,1))$ and $u(t)$ is an unknown function.

Let us approximate $f(t), u(t)$, and $k_1(t,s)$ by using the collocation points t_i as given in the above section 2.2. Then the numerical procedure as follows:

STEP 1: Let us first approximate $f(t) \approx X^T\Psi(t)$, and $u(t) \approx Y^T\Psi(t)$, (3.2)

Let the function $f(t) \in L^2[0,1]$ may be expanded as:

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} x_{n,m} L_{n,m}(t), \tag{3.3}$$

where

$$x_{n,m} = (f(t), L_{n,m}(t)). \tag{3.4}$$

In (3.4), (. , .) denotes the inner product.

If the infinite series in (3.3) is truncated, then (3.3) can be rewritten as:

$$f(t) = \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} x_{n,m} L_{n,m}(t) = X^T\Psi(t), \tag{3.5}$$

where X and $\Psi(t)$ are $N \times 1$ matrices given by:

$$\begin{aligned} X &= [x_{10}, x_{11}, \ldots, x_{1,M-1}, x_{20}, \ldots, x_{2,M-1}, \ldots, x_{2^{k-1},0}, \ldots, x_{2^{k-1},M-1}]^T \\ &= [x_1, x_2, \ldots, x_{2^{k-1}M}]^T, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \Psi(t) &= [L_{10}(t), L_{11}(t), \ldots, L_{1,M-1}(t), L_{20}(t), \ldots, L_{2,M-1}(t), \ldots, L_{2^{k-1},0}(t), \ldots, L_{2^{k-1},M-1}(t)]^T \\ &= [L_1(t), L_2(t), \ldots, L_{2^{k-1}M}(t)]^T. \end{aligned} \tag{3.7}$$

STEP 2: Next, approximate the kernel function as: $k_1(t,s) \in L^2([0,1]\times[0,1])$

$$k_1(t,s) \approx \Psi^T(t) K_1 \Psi(s), \tag{3.8}$$

where K_1 is $2^{k-1}M \times 2^{k-1}M$ matrix, with

$$[K_1]_{ij} = (L_i(t), (k_1(t,s), L_j(s))).$$

$$\text{i.e., } K_1 \approx \left[\Psi^T(t)\right]^{-1} \cdot \left[k_1(t,s)\right] \cdot \left[\Psi(s)\right]^{-1} \tag{3.9}$$

STEP 3: Substituting Eq. (3.2) and Eq. (3.8) in Eq. (3.1), we have:

$$\Psi^T(t)Y = \Psi^T(t)X + \int_0^1 \Psi^T(t)K_1\Psi(s)\Psi^T(s)Y ds$$

$$\Psi^T(t)Y = \Psi^T(t)X + \Psi^T(t)K_1\left(\int_0^1 \Psi(s)\Psi^T(s)\, ds\right)Y$$

$$\Psi^T(t)Y = \Psi^T(t)(X + K_1Y),$$

Then we get a system of equations as,

$$(I - K_1)Y = X. \tag{3.10}$$

By solving this system obtain the Legendre wavelet coefficients 'Y' and substituting in step 4.

STEP 4: $u(t) \approx Y^T\Psi(t)$

This is the required approximate solution of Eq. (3.1).

Volterra Integral equations:

Consider the Volterra integral equations with convolution but non-symmetrical kernel

$$u(t) = f(t) + \int_0^t k_2(t,s)\, u(s)\, ds, \quad t \in [0,1] \tag{3.11}$$

where $f(t) \in L^2[0,1)$, $k_2(t,s) \in L^2([0,1)\times[0,1))$ and $u(t)$ is an unknown function.

Let us approximate $f(t), u(t)$, and $k_2(t,s)$ by using the collocation points t_i as given in the above section 2.2. Then the numerical procedure as follows:

STEP 1: This can be rewritten in Fredholm integral equations, with a modified kernel $\tilde{k}_2(t,s)$ and solved in Fredholm form [22] as,

$$u(t) = f(t) + \int_0^t \tilde{k}_2(t,s)\, u(s)\, ds, \tag{3.12}$$

where, $\tilde{k}_2(t,s) = \begin{cases} k_2(t,s), & 0 \le s \le t \\ 0, & t \le s \le 1. \end{cases}$

STEP 2: Let us first approximate $f(t)$ and $u(t)$ as given in Eq. (3.2),

STEP 3: Next, we approximate the kernel function as: $\tilde{k}_2(t,s) \in L^2([0,1]\times[0,1])$

$$\tilde{k}_2(t,s) \approx \Psi^T(t) \cdot K_2 \cdot \Psi(s), \tag{3.13}$$

where K_2 is $2^{k-1}M \times 2^{k-1}M$ matrix, with

$$(K_2)_{ij} = (L_i(t), (\tilde{k}_2(t,s), L_j(s))).$$

$$\text{i.e., } K_2 \approx \left[\Psi^T(t)\right]^{-1} \cdot \left[\tilde{k}_2(t,s)\right] \cdot \left[\Psi(s)\right]^{-1} \tag{3.14}$$

STEP 4: Substituting Eq. (3.2) and Eq. (3.13) in Eq. (3.12), we have:

$$\Psi^T(t)Y = \Psi^T(t)X + \int_0^1 \Psi^T(t) K_2 \Psi(s) \Psi^T(s) Y ds$$

$$\Psi^T(t)Y = \Psi^T(t)X + \Psi^T(t) K_2 \left(\int_0^1 \Psi(s)\Psi^T(s)\, ds\right) Y$$

$$\Psi^T(t)Y = \Psi^T(t)(X + K_2 Y),$$

Then we get a system of equations as,

$$(I - K_2)Y = X. \tag{3.15}$$

By solving this system obtain the Legendre wavelet coefficients '*Y*' and substituting in step 5.

STEP 5: $u(t) \approx Y^T \Psi(t)$

This is the required approximate solution of Eq. (3.11).

Fredholm-Volterra integral equations:

Consider the Fredholm-Volterra integral equation of the second kind,

$$u(t) = f(t) + \int_0^1 k_1(t,s)u(s)\,ds + \int_0^x k_2(t,s)u(s)\,ds, \qquad (3.16)$$

where $f(t) \in L^2[0,1)$, $k_1(t,s)$ and $k_2(t,s) \in L^2([0,1)\times[0,1))$ are known function and $u(t)$ is an unknown function.

Let us approximate $f(t), u(t), k_1(t,s)$ and $k_2(t,s)$ by using the collocation points as follows:

STEP 1: Let us first approximate $f(t)$ and $u(t)$ as given in Eq. (3.2),

STEP 2: Substituting Eq. (3.2), Eq. (3.9) and Eq. (3.14) in Eq. (3.16), we get a system of N equations with N unknowns,

$$\text{i.e., } (I - K_1 - K_2)Y = X. \qquad (3.17)$$

where, I is an identity matrix.

By solving this system we obtain the Legendre wavelet coefficient 'Y' and substituting this 'Y' in step 3.

STEP 3: $u(t) \approx Y^T\Psi(t)$

This is the required approximate solution of Eq. (3.16).

Weakly singular Fredholm integral equations:

Consider the Weakly singular Fredholm integral equation,

$$u(t) = f(t) + \int_0^1 \frac{u(s)}{\sqrt{1-t}}\,ds, \quad 0 \le t\ s \le 1 \qquad (3.18)$$

To solve Eq. (3.18), the procedure as follows:

STEP 1: We first approximate $u(t)$ as truncated series defined in Eq. (3.5). That is,

$$u(t) = Y^T\Psi(t) \qquad (3.19)$$

where Y and $\Psi(t)$ are defined similarly to Eqs. (3.6) and (3.7).

STEP 2: Then substituting Eq. (3.19) in Eq. (3.18), we get,

$$Y^T\Psi(t) = f(t) + \int_0^1 \frac{Y^T\Psi(s)}{\sqrt{1-t}}\,ds \qquad (3.20)$$

STEP 3: Substituting the collocation point t_i in Eq. (3.20). We obtain,

$$Y^T\Psi(t_i) = f(t_i) + \int_0^1 \frac{Y^T\Psi(s)}{\sqrt{1-t_i}}\,ds \qquad (3.21)$$

$$Y^T(\Psi(t_i) - G) = f, \text{ where } G = \int_0^1 \frac{Y^T\Psi(s)}{\sqrt{1-t_i}}\,ds$$

STEP 4: Now, we get the system of algebraic equations with unknown coefficients.

$$Y^T K = f, \text{ where } K = (\Psi(x_i) - G)$$

Solving the above system of equations, we get the Legendre wavelet coefficients 'Y' and then substituting these coefficients in Eq. (3.19), we obtain the required approximate solution of Eq. (3.18).

3.2 Integro-differential Equations

Fredholm Integro-differential equations:

In this section, we concerned about a technique that will reduce Fredholm integro-differential equation to an equivalent Fredholm integral equation. This can be easily done by integrating both sides of the integro-differential equation as many times as the order of the derivative involved in the equation from 0 to t for every time we integrate, and using the given initial conditions. It is worth noting that this method is applicable only if the Fredholm integro-differential equation involves the unknown function $u(t)$ only, and not any of its derivatives, under the integral sign [1].

Consider the Fredholm integro-differential equation,

$$u^{(n)}(t) = f(t) + \int_0^1 k_1(t,s)\,u(s)\,ds\,,\ u^{(l)} = b_l, \tag{3.22}$$

where $f(t) \in L^2[0,1),\ k_1(t,s) \in L^2([0,1)\times[0,1))$ and $u^{(n)}(t)$ is an unknown function.

where $u^{(n)}(t)$ is the n^{th} derivative of $u(t)$ with respect to t and b_l are constants that define the initial conditions.

Let us first, we convert the Fredholm integro-differential equation into Fredholm integral equation, then we reduce it into a system of algebraic equations as given in Eq. (3.10), using this system we solve the Eq. (3.22). Then we obtain the approximate solution of equation.

Volterra Integro-differential equations:

In this section, we concerned with converting to Volterra integral equations. We can easily convert the Volterra integro-differential equation to equivalent Volterra integral equation, provided the kernel is a difference kernel defined by $k(t, s) = k(t - s)$. This can be easily done by integrating both sides of the equation and using the initial conditions. To perform the conversion to a regular Volterra integral equation, we should use the well-known formula, which converts multiple integrals into a single integral [1].

i.e.,

$$\int_0^t \int_0^t \ldots\ldots\int_0^t u(t)dt^n = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}u(s)ds$$

Consider the Volterra integro-differential equations,

$$u^{(n)}(t) = f(t) + \int_0^t k_2(t,s)\,u(s)\,ds\,,\ u^{(l)} = b_l, \tag{3.22}$$

where $f(t) \in L^2[0,1),\ k_2(t,s) \in L^2([0,1)\times[0,1))$ and $u^{(n)}(t)$ is an unknown function.

where $u^{(n)}(t)$ is the n^{th} derivative of $u(t)$ with respect to t and b_l are constants that define the initial conditions.

Let us first, we convert the Volterra integro-differential equation into Volterra integral equation, then we reduce it into a system of algebraic equations as given in Eq. (3.15), using this system we solve the Eq. (3.22). Then we obtain the approximate solution of equation.

4. Convergence Analysis

Theorem: The series solution $u(t)=\sum_{p=1}^{\infty}\sum_{q=0}^{\infty}x_{p,q}L_{p,q}(t)$ defined in Eq. (3.5) using Legendre wavelet method converges to $u(t)$ as given in [23].

Proof: Let $L^2(R)$ be the Hilbert space and $L_{p,q}$ defined in Eq. (3.2) forms an orthonormal basis.

Let $u(t)=\sum_{i=0}^{M-1}x_{p,i}L_{p,i}(t)$ where $x_{p,i}=\langle u(t),L_{p,i}(t)\rangle$ for a fixed p.

Let us denote $L_{p,i}(t)=L(t)$ and let $\alpha_j=\langle u(t),L(t)\rangle$.

Now we define the sequence of partial sums S_p of $(\alpha_j L(t_j))$; Let S_p and S_q be the partial sums with $p\geq q$. We have to prove S_p is a Cauchy sequence in Hilbert space.

Let $S_p=\sum_{i=1}^{p}\alpha_j L(t_j)$.

$$\text{Now } \langle u(t),S_p\rangle=\left\langle u(t),\sum_{i=1}^{p}\alpha_j L(t_j)\right\rangle=\sum_{j=1}^{p}|\alpha_j|^2.$$

We claim that $\left\|S_p-S_q\right\|^2=\sum_{j=q+1}^{p}|\alpha_j|^2,\quad p>q.$

Now

$$\left\|\sum_{j=q+1}^{p}\alpha_j L(t_j)\right\|^2=\left\langle\sum_{j=q+1}^{p}\alpha_j L(t_j),\sum_{j=q+1}^{p}\alpha_j L(t_j)\right\rangle=\sum_{j=q+1}^{p}|\alpha_j|^2,\ \ for\ \ p>q.$$

Therefore, $\left\|\sum_{j=q+1}^{p}\alpha_j L(t_j)\right\|^2=\sum_{j=1}^{p}|\alpha_j|^2,\ \ for\ \ p>q.$

From Bessel's inequality, we have $\sum_{j=1}^{p}|\alpha_j|^2$ is convergent and hence

$$\left\|\sum_{j=q+1}^{p}\alpha_j L(t_j)\right\|^2\to 0\quad as\ \ q,p\to\infty$$

So, $\left\|\sum_{j=q+1}^{p}\alpha_j L(t_j)\right\|\to 0$ and $\{S_p\}$ is a Cauchy sequence and it converges to s (say).

We assert that $u(t)=s$,

Now $\left\langle s-u(t), L(t_j)\right\rangle = \left\langle s, L(t_j)\right\rangle - \left\langle u(t), L(t_j)\right\rangle = \left\langle \lim_{p\to\infty} S_p, L(t_j)\right\rangle - \alpha_j = \alpha_j - \alpha_j$,

This implies,

$$\left\langle s-u(t), L(t_j)\right\rangle = 0$$

Hence $u(t)=s$ and $\sum_{i=1}^{p} \alpha_j L(t_j)$. converges to $u(t)$ as $p\to\infty$ and proved.

5. Numerical experiments

In this section, we present Legendre wavelet (LW) collocation method for the numerical solution of integral and integro-differential equation in comparison with existing method to demonstrate the capability of the proposed method and error analysis are shown in tables and figures. Error function is presented to verify the accuracy and efficiency of the following numerical results:

$$E_{\max} = Error\ function = \left\| u_e(t_i) - u_a(t_i)\right\|_\infty = \sqrt{\sum_{i=1}^{n}\left(u_e(t_i)-u_a(t_i)\right)^2}$$

where u_e and u_a are the exact and approximate solution respectively.

Example 5.1 Let us consider the Fredholm integral equation of the second kind [4].

$$u(t) = t^2 + \int_0^1 (t+s)\ u(s)\ ds \tag{5.1}$$

which has the exact solution $u(t)=t^2-5t-17/6$. Where $f(t)=t^2$ and kernel $k_1(t,s)=t+s$.

Firstly, we approximate $f(t) \approx X^T\Psi(t)$, and $u(t)\approx Y^T\Psi(t)$,

Next, approximate the kernel function as: $k_1(t,s)\in L^2([0,1]\times[0,1])$

$$k_1(t,s) \approx \Psi^T(t)K_1\Psi(s),$$

where K_1 is $2^{k-1}M\times 2^{k-1}M$ matrix, with $[K_1]_{ij} = (H_i(t),(k_1(t,s),H_j(s)))$.

$$K_1 \approx \left[\Psi^T(t)\right]^{-1}\cdot\left[k_1(t,s)\right]\cdot\left[\Psi(s)\right]^{-1}$$

Next, substituting the function $f(t)$, $u(t)$, and $k_1(t,s)$ in Eq. (5.1), then using the collocation points, we get the system of algebraic equations with unknown coefficients for k = 2 and M = 4 (N = 8), as an order 8×8 as follows:

$$\Psi^T(t)Y = \Psi^T(t)X + \int_0^1 \Psi^T(t)K_1\Psi(s)\Psi^T(s)Yds$$

$$\Psi^T(t)Y = \Psi^T(t)X + \Psi^T(t)K_1\left(\int_0^1 \Psi(s)\Psi^T(s)\,ds\right)Y$$

$$\Psi^T(t)Y = \Psi^T(t)(X+K_1Y),$$

$$(I-K_1)Y = X, \quad \text{where, } I = \int_0^1 \Psi(s)\Psi^T(s)\,ds \text{ is the identity matrix.}$$

where, $X = [0.0589 \quad 0.0510 \quad 0.0132 \quad 0 \quad 0.4125 \quad 0.1531 \quad 0.0132 \quad 0]$,

$$K_1 = \begin{bmatrix} 0.2500 & 0.0722 & 0.0000 & 0 & 0.5000 & 0.0722 & 0.0000 & 0.0000 \\ 0.0722 & 0.0000 & 0.0000 & -0.0000 & 0.0722 & 0.0000 & 0 & 0.0000 \\ 0.0000 & 0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 \\ 0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000 \\ 0.5000 & 0.0722 & 0.0000 & 0.0000 & 0.7500 & 0.0722 & 0.0000 & 0.0000 \\ 0.0722 & 0.0000 & 0 & 0.0000 & 0.0722 & 0.0000 & 0.0000 & -0.0000 \\ 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 \\ 0.0000 & 0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.0000 \end{bmatrix}$$

By solving this system of equations, we obtain the Legendre wavelet coefficients,

$Y = [-2.8284 \quad -0.4593 \quad 0.0132 \quad -0.0000 \quad -4.2426 \quad -0.3572 \quad 0.0132 \quad -0.0000]$

and substituting these coefficients in $u(t) = Y^T\Psi(t)$, we get the approximate solution $u(t)$ are shown in table 1. Maximum Error analysis is shown in table 2 and compared with existing method (Haar wavelet).

Table 1: Numerical results of the example 5.1.

t	Exact	Legendre Wavelet
0.0625	-3.1419	-3.1419
0.1875	-3.7357	-3.7357
0.3125	-4.2982	-4.2982
0.4375	-4.8294	-4.8294
0.5625	-5.3294	-5.3294
0.6875	-5.7982	-5.7982
0.8125	-6.2357	-6.2357
0.9375	-6.6419	-6.6419

Table 2: Maximum Error analysis ($E_{\max}$) of the example 5.1.

N	Method (Lepik and Tamme (2005b))	LW
8	7.0e-02	2.7e-14
16	1.7e-02	1.1e-14
32	4.3e-03	1.5e-14
64	1.3e-03	2.1e-14

Example 5.2 Next, consider the Fredholm integral equation,

$$u(t) = e^{2t+(1/3)} + \int_0^1 \left(-\frac{1}{3}e^{2t-(5/3)s}\right)u(s)\,ds \tag{5.2}$$

with exact solution $u(t) = e^{2t}$. Applying the present method and solved the Eq. (5.2), numerical results are presented in table 3 and figure 1 in comparison with existing method [24].

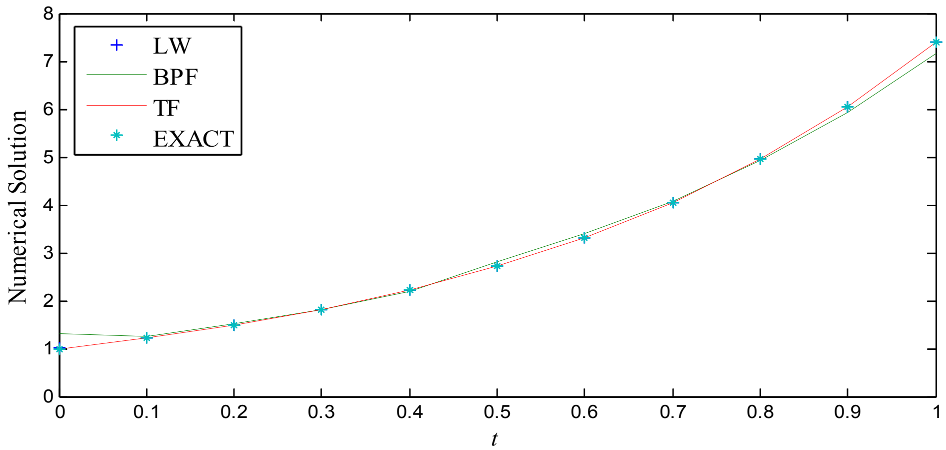

Fig. 1: Comparison of LW, BPF and TF with exact solutions.

Table 3: Numerical results is obtained for $M = 8$ and $k = 3$ of the example 5.2.

t	Exact	BPF	TF	LW
0	1	1.301832	0.999844	1.000005
0.1	1.221403	1.244627	1.221598	1.221410
0.2	1.491825	1.501307	1.492294	1.497834
0.3	1.822119	1.810922	1.822684	1.822130
0.4	2.225541	2.184388	2.225880	2.225554
0.5	2.718282	2.804810	2.717857	2.718298
0.6	3.320117	3.383247	3.320648	3.320137
0.7	4.055200	4.080975	4.056474	4.055224
0.8	4.953032	4.922595	4.954570	4.953062
0.9	6.049647	5.937783	6.050568	6.049683
1	7.389056	7.162334	7.387901	7.389100

Example 5.3 Next, consider [15],

$$u(t) = \sin(2\pi t) + \int_0^1 \cos(t)\, u(s)\, ds\,. \tag{5.3}$$

which has the exact solution of the form $u(t) = \sin(2\pi t)$. Solving Eq. (5.3) using the present method, we get the approximate solution of $u(t)$ with the help of Legendre wavelet coefficients. Error analysis is compared with existing method are shown in table 4.

Example 5.4 Next, consider [15],

$$u(t) = \sin(2\pi t) + \int_0^1 (t^2 - t - s^2 + s)\, u(s)\, ds \tag{5.4}$$

which has the exact solution of the form $u(t) = \sin(2\pi t)$. Solving Eq. (5.4) using the present method, we get the approximate solution of $u(t)$ with the help of Legendre wavelet coefficients. Error analysis is compared with existing method are shown in table 4.

Example 5.5 Next, consider [15],

$$u(t) = -2t^3 + 3t^2 - t + \int_0^1 (t^2 - t - s^2 + s)\, u(s)\, ds \tag{5.5}$$

which has the exact solution of the form $u(t) = -2t^3 + 3t^2 - t$. Solving Eq. (5.5) using the present method, we get the approximate solution of $u(t)$ with the help of Legendre wavelet coefficients. Error analysis is compared with existing method are shown in table 4.

Table 4: Comparison of the Error analysis.

N	Example 5.3		Example 5.4		Example 5.5	
	Method [15]	$E_{Max}(LW)$	Method [15]	$E_{Max}(LW)$	Method [15]	$E_{Max}(LW)$
4	2.84e-02	3.33e-16	2.84e-02	2.22e-16	1.33e-10	0
8	2.38e-03	1.60e-15	2.38e-03	5.55e-16	3.79e-10	4.85e-17
16	2.09e-04	1.22e-15	2.10e-04	8.88e-16	3.26e-10	1.80e-16
32	1.20e-04	1.54e-15	2.00e-04	1.66e-15	4.83e-10	1.38e-16

Example 5.6 Let us consider the Volterra integral equation of the second kind [22],

$$u(t) = \cos(t) - \int_0^t (t-s)\cos(t-s)\, u(s)\, ds, \qquad 0 \le t \le 1 \tag{5.6}$$

which has the exact solution $u(t) = \frac{1}{3}(2\cos\sqrt{3}t + 1)$. Where $f(t) = \cos(t)$ and kernel $k_2(t,s) = -(t-s)\cos(t-s)$.

Firstly, we approximate $f(t) \approx X^T \Psi(t)$, and $u(t) \approx Y^T \Psi(t)$,

Next, approximate the kernel function as: $k_2(t,s) \in L^2([0,1]\times[0,1])$

$$k_2(t,s) \approx \Psi^T(t) K_2 \Psi(s),$$

where K_2 is $2^{k-1}M \times 2^{k-1}M$ matrix, with $[K_2]_{ij} = (H_i(t), (k_2(t,s), H_j(s)))$.

$$K_2 \approx \left[\Psi^T(t)\right]^{-1} \cdot \left[k_2(t,s)\right] \cdot \left[\Psi(s)\right]^{-1}$$

Next, substituting the $f(t), u(t)$, and $k_2(t,s)$ in Eq. (5.6) using the collocation points, we get the system of algebraic equations with unknown coefficients for k = 2 and M = 4 (N = 8), as an order 8×8 as follows:

$$\Psi^T(t)Y = \Psi^T(t)X + \int_0^1 \Psi^T(t) K_2 \Psi(s) \Psi^T(s) Y ds$$

$$\Psi^T(t)Y = \Psi^T(t)X + \Psi^T(t) K_2 \left(\int_0^1 \Psi(s)\Psi^T(s)\, ds \right) Y$$

$$\Psi^T(t)Y = \Psi^T(t)(X + K_2 Y),$$

$$(I-K_2)Y = X, \quad \text{where, } I=\int_0^1 \Psi(s)\Psi^T(s)\,ds \text{ is the identity matrix.}$$

where, $X=[0.6780 \quad -0.0251 \quad -0.0064 \quad 0.0001 \quad 0.5120 \quad -0.0691 \quad -0.0048 \quad 0.0002]$,

$$K_2=\begin{bmatrix} -0.0388 & 0.0343 & -0.0076 & -0.0003 & 0 & 0 & 0 & 0 \\ -0.0343 & 0.0261 & 0.0010 & 0.0020 & 0 & 0 & 0 & 0 \\ -0.0076 & -0.0010 & 0.0115 & 0.0000 & 0 & 0 & 0 & 0 \\ 0.0003 & 0.0020 & -0.0000 & 0.0154 & 0 & 0 & 0 & 0 \\ -0.2050 & 0.0432 & 0.0064 & -0.0005 & -0.0388 & 0.0343 & -0.0076 & -0.0003 \\ -0.0432 & -0.0143 & 0.0016 & 0.0001 & -0.0343 & 0.0261 & 0.0010 & 0.0020 \\ 0.0064 & -0.0016 & -0.0001 & 0.0000 & -0.0076 & -0.0010 & 0.0115 & 0.0000 \\ 0.0005 & 0.0001 & -0.0000 & -0.0000 & 0.0003 & 0.0020 & -0.0000 & 0.0154 \end{bmatrix}$$

By solving this system of equations, we obtain the Legendre wavelet coefficients 'Y'

$Y=[0.6512 \quad -0.0487 \quad -0.0114 \quad 0.0002 \quad 0.3586 \quad -0.1118 \quad -0.0032 \quad 0.0004]$ and substituting these coefficients in $u(t)=Y^T\Psi(t)$, we get the approximate solution $u(t)$ as shown in table 5. Maximum error analysis is compared with Hermite wavelet is shown in table 6.

Table 5: Numerical results of the example 5.6.

t	Exact	Legendre Wavelet
0.0625	0.9961	0.9980
0.1875	0.9652	0.9655
0.3125	0.9047	0.9055
0.4375	0.8176	0.8190
0.5625	0.7078	0.7091
0.6875	0.5806	0.5802
0.8125	0.4419	0.4424
0.9375	0.2980	0.2981

Table 6: Comparison of maximum error analysis E_{max} of the example 5.6.

N	HW	LW
8	1.52e-02	1.94e-03
16	4.00e-03	4.88e-04
32	1.00e-03	1.22e-04
64	2.49e-04	3.05e-05
128	6.23e-05	7.62e-06

Example 5.7 Next, consider the Fredholm integro-differential equation [1],

$$u'(t)=36t^2+\int_0^1 u(s)\,ds, \; u(0)=1, \quad 0\le t\le 1 \tag{5.7}$$

which has the exact solution $u(t)=1+8t+12t^3$.

Firstly, integrating Eq. (5.7) w.r.t t, we get the Fredholm integral equation,

$$u(t) = 1 + 12t^3 + t\int_0^1 u(s)\,ds, \tag{5.8}$$

Solving Eq. (5.8) applying the present method, we obtain the approximate solution of $u(t)$ with the help of Legendre wavelet coefficients. Maximum error analysis is shown in table 7 compared with Hermite wavelet.

Table 7: Maximum error analysis of the example 5.7.

N	HW	LW
8	3.51e-01	1.06e-14
16	9.08e-02	3.55e-15
32	2.30e-02	1.06e-14
64	5.81e-03	3.55e-15
128	1.45e-03	2.13e-14

Example 5.8 Next, consider the Volterra integro-differential equation [25],

$$u'(t) = 1 - 2t\sin t + \int_0^t u(s)\,ds,\ u(0) = 0, \quad 0 \le t \le 1 \tag{5.9}$$

which has the exact solution $u(t) = t\cos t$.

Firstly, integrating Eq. (5.9) w.r.t t, we get Volterra integral equation,

$$u(t) = t - 2(\sin t - t\cos t) + \int_0^t (t-s)\,u(s)\,ds, \tag{5.10}$$

Solving Eq. (5.10) using the present method, we get the approximate solution of $u(t)$ with the help of Legendre wavelet coefficients. Maximum error analysis is shown in table 8 compared with Hermite wavelet.

Table 8: Maximum error analysis of the example 5.8.

N	HW	LW
8	1.37e-02	1.21e-03
16	3.49e-03	3.10e-04
32	8.81e-04	7.82e-05
64	2.20e-04	1.96e-05
128	5.51e-05	4.92e-06

Example 5.9 Next, consider the Volterra-Fredholm integral equation [1],

$$u(t) = \exp(t) + 1 + t + \int_0^t (t-s)\,u(s)\,ds - \int_0^1 \exp(t-s)\,u(s)\,ds, \quad 0 \le t \le 1 \tag{5.11}$$

which has the exact solution $u(t)=\exp(t)$. Where $f(t)=\exp(t)+1+t$ and the kernels $k_1(t,s)=-\exp(t-s)$ and $k_2(t,s)=(t-s)$.

Let us approximate $f(t), u(t)$, $k_1(t,s)$ and $k_2(t,s)$ as given in Eq. (3.5), Eq. (3.9) and Eq. (3.14) using the collocation points, we get an system of N equations with N unknowns

i.e., $(I-K_1-K_2)Y=X$. where, I is an identity matrix,

we find, X = [1.8013 0.2339 0.0085 0.0004 2.7500 0.3195 0.0140 0.0006],

$$K_1=\begin{bmatrix} -0.5105 & 0.0734 & -0.0047 & 0.0002 & -0.3096 & 0.0445 & -0.0029 & 0.0001 \\ -0.0734 & 0.0105 & -0.0007 & 0.0000 & -0.0445 & 0.0064 & -0.0004 & 0.0000 \\ -0.0047 & 0.0007 & -0.0000 & 0.0000 & -0.0029 & 0.0004 & -0.0000 & 0.0000 \\ -0.0002 & 0.0000 & -0.0000 & 0.0000 & -0.0001 & 0.0000 & -0.0000 & 0.0000 \\ -0.8417 & 0.1210 & -0.0078 & 0.0003 & -0.5105 & 0.0734 & -0.0047 & 0.0002 \\ -0.1210 & 0.0174 & -0.0011 & 0.0000 & -0.0734 & 0.0105 & -0.0007 & 0.0000 \\ -0.0078 & 0.0011 & -0.0001 & 0.0000 & -0.0047 & 0.0007 & -0.0000 & 0.0000 \\ -0.0003 & 0.0000 & -0.0000 & 0.0000 & -0.0002 & 0.0000 & -0.0000 & 0.0000 \end{bmatrix}$$

$$K_2=\begin{bmatrix} 0.0403 & -0.0361 & 0.0085 & 0.0000 & 0 & 0 & 0 & 0 \\ 0.0361 & -0.0281 & 0.0000 & -0.0022 & 0 & 0 & 0 & 0 \\ 0.0085 & -0.0000 & -0.0111 & 0.0000 & 0 & 0 & 0 & 0 \\ 0.0000 & -0.0022 & -0.0000 & -0.0152 & 0 & 0 & 0 & 0 \\ 0.2500 & -0.0722 & 0.0000 & 0.0000 & 0.0403 & -0.0361 & 0.0085 & 0.0000 \\ 0.0722 & 0.0000 & 0.0000 & 0.0000 & 0.0361 & -0.0281 & 0.0000 & -0.0022 \\ 0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0085 & -0.0000 & -0.0111 & 0.0000 \\ 0.0000 & 0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0022 & -0.0000 & -0.0152 \end{bmatrix}$$

By solving this system we obtain the Legendre wavelet coefficient,

Y = [0.9168 0.1315 0.0077 -0.0003 1.5114 0.2168 0.0128 -0.0005],

Then, substituting these coefficients in $u(t) \cong Y^T\Psi(t)$, we get the approximate solution of Eq. (5.11) are shown in table 9. Maximum error analysis is shown in table 10 compared with Hermite wavelet.

Table 9: Numerical results of the example 5.9.

t	Exact	Legendre Wavelet
0.0625	1.0645	1.0632
0.1875	1.2062	1.2057
0.3125	1.3668	1.3675
0.4375	1.5488	1.5467
0.5625	1.7551	1.7529
0.6875	1.9887	1.9877
0.8125	2.2535	2.2544
0.9375	2.5536	2.5498

Table 10: Maximum error analysis of the example 5.9.

N	HW	LW
4	1.30e-02	6.12e-03
8	2.18e-02	3.77e-03
16	5.76e-03	9.56e-04
32	1.48e-03	2.40e-04
64	3.77e-04	6.02e-05

Example 5.10 Let us consider the weakly singular Fredholm integral equation of the second kind,

$$u(t) = t^2 - \frac{16}{15} + \int_0^1 \frac{u(s)}{\sqrt{1-t}} ds, \quad 0 \le t \le 1. \tag{5.12}$$

which has the exact solution $u(t) = t^2$. Applying the Legendre Wavelet Collocation Method, we solved the Eq. (5.12) with $k = 1$ and $M = 3$, then we obtain,

$$f = [-1.0389 \quad -0.8167 \quad -0.3722]$$

$$K = \begin{bmatrix} -1.0000 & -1.0000 & -1.0000 \\ -2.3094 & -1.1547 & 0.0000 \\ -0.5217 & -2.0125 & -0.5217 \end{bmatrix}$$

Next, we get the Legendre wavelet coefficients,

$$Y = \begin{bmatrix} 0.3333 & 0.2887 & 0.0745 \end{bmatrix}$$

next, substituting these coefficients in Eq. (3.19), we get the accurate solution of Eq. (5.12) with exact solution $u(t) = t^2$ and the maximum error is 1.11e-016 compared to the existing method (Behzadi (2014)) has the maximum error for n = 256 is 5.02e-06. This shows the efficiency of the proposed method.

Example 5.11 Next, consider [26],

$$u(t) = \sqrt{t} - \frac{\pi}{2} + \int_0^1 \frac{u(s)}{\sqrt{1-t}} ds, \quad 0 \le t \le 1. \tag{5.13}$$

which has the exact solution $u(t) = \sqrt{t}$. We solved the Eq. (5.13) by approaching the present method for $k = 1$ and $M = 5$, we get the approximate solution as shown in table 5.11 and the maximum error is 2.98e-04.

Table 5.11 Numerical result of the example 5.11.

t	Exact solution	Legendre wavelet ($k = 1$, $M = 5$)	Absolute Error
0.1	0.3162	0.3159	2.98e-04
0.2	0.4472	0.4445	2.70e-03
0.3	0.5477	0.5474	2.98e-04
0.4	0.6325	0.6328	3.29e-04
0.5	0.7071	0.7068	2.98e-04
0.6	0.7746	0.7738	7.54e-04
0.7	0.8367	0.8364	2.98e-04
0.8	0.8944	0.8950	5.39e-04
0.9	0.9487	0.9484	2.98e-04

Example 5.12 Lastly, consider [26],

$$u(t) = \exp(t) - 4.0602 + \int_0^1 \frac{u(s)}{\sqrt{1-t}} ds, \quad 0 \le t \le 1. \tag{5.14}$$

which has the exact solution $u(t) = \exp(t)$. Applying the proposed method to solve Eq. (5.14) for $k = 1$ and $M = 8$. We obtain the approximate solution $u(t)$ as shown in table 12 and the maximum error is 4.30e-05 as shown in figure 2.

Table 12: Numerical result of the example 5.12.

t	Exact solution	Present method ($k = 1$, $M = 8$)
0.1	1.105170	1.105213
0.2	1.221402	1.221445
0.3	1.349858	1.349901
0.4	1.491824	1.491867
0.5	1.648721	1.648764
0.6	1.822118	1.822161
0.7	2.013752	2.013795
0.8	2.225540	2.225583
0.9	2.459603	2.459646

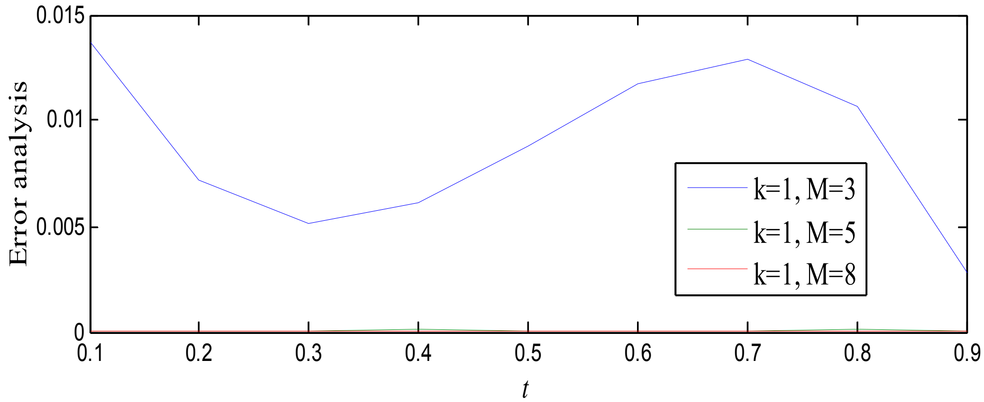

Fig. 2: Error analysis of the example 5.12.

6. CONCLUSION

In this paper, we proposed the Legendre wavelet collocation method for solving the integral and integro-differential equations. The proposed scheme reduces an integral equation into a set of algebraic equations. The numerical result shows that the accuracy improves with increasing the level of resolution *N*, for better accuracy, the larger *N* is recommended. Error analysis is presented in comparison with the existing methods as shown in tables and figures. Hence the present scheme shows the efficiency of the Legendre wavelet collocation method.

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References

[1] Wazwaz A. M., Linear and nonlinear integral equations methods and applications, Springer, 2011.

[2] Chui C. K., Wavelets: A mathematical tool for signal analysis, in: SIAM Monographs on Mathematical Modeling and Computation, SIAM, Philadelphia, Pennsylvania, 1997.

[3] Beylkin G., Coifman R. and Rokhlin V., Fast wavelet transforms and numerical algorithms I. Commun. Pure Appl. Math., 44, 1991, 141–183.

[4] Lepik Ü. and Tamme E., Application of the Haar wavelets for solution of linear integral equations, Ant. Turk–Dynam. Sys. Appl. Proce., 2005, 395–407.

[5] Lepik Ü., Numerical solution of differential equations using Haar wavelets, Math. Comput. Simul., 68, 2005, 127-143.

[6] Lepik Ü. and Tamme E., Application of the Haar wavelets for solution of linear integral equations, Ant. Turk–Dynam. Sys. Appl. Proce., 2005, 395–407.

[7] Lepik Ü., Haar wavelet method for nonlinear integro-differential equations, Appl. Math. Comp., 176, 2006, 324-333.

[8] Lepik Ü., Application of the Haar wavelet transform to solving integral and differential Equations, Proc. Estonian Acad Sci. Phys. Math., 56, 2007, 28-46.

[9] Lepik Ü. and Tamme E., Solution of nonlinear Fredholm integral equations via the Haar wavelet method, Proc. Estonian Acad. Sci. Phys. Math., 56, 2007, 17–27.

[10] Lepik Ü., Numerical solution of evolution equations by the Haar wavelet method, Appl. Math. Comput. 185, 2007, 695–704.

[11] Lepik Ü., Solving fractional integral equations by the Haar wavelet method, Appl. Math. Comp., 214, 2009, 468-478.

[12] Maleknejad K. and Mirzaee F., Numerical solution of linear Fredholm integral equations system by rationalized Haar functions method, 80(11), 2003, 1397-1405.

[13] Maleknejad K. and Mirzaee F., Using rationalized haar wavelet for solving linear integral equations, App. Math. Comp., 160, 2005, 579 – 587.

[14] Maleknejad K., Kajani M. T. and Mahmoudi Y., Numerical solution of linear Fredholm and Volterra integral equation of the second kind by using Legendre wavelets, J. Kybernet., 32, 2003, 1530-1539.

[15] Maleknejad K. and Yousefi M., Numerical solution of the integral equation of the second kind by using wavelet bases of Hermite cubic splines, App. Math. Comp., 183, 2006, 134-141.

[16] Maleknejad K., Lotfi T. and Rostami Y., Numerical computational method in solving Fredholm integral equations of the second kind by using Coifman wavelet, App. Math. Comp. 186, 2007, 212-218.

[17] Babolian E. and Fattahzadeh F., Numerical computational method in solving integral equations by using chebyshev wavelet operational matrix of integration, Appl. Math. Comp., 188, 2007, 1016 - 1022.

[18] Abdalrehman A. A., An Algorithm for nth Order Intgro-Differential Equations by Using Hermite Wavelets Functions, Bagh. Sci. Jour., 11(3) 2014.

[19] Yousefi S. and Banifatemi A., Numerical solution of Fredholm integral equations by using CAS wavelets, App. Math. Comp., 183, 2006, 458-463.

[20] Yousefi S. and Razzaghi M., Legendre wavelets method for the nonlinear Volterra-Fredholm integral equations, Math. Comp. Simul., 70, 2005, 1-8.

[21] Yousefi S. A., Numerical solution of Abel's integral equation by using Legendre wavelets. Appl Math Comput 175, 2006, 574–80.

[22] Blyth W. F., May R. L. and Widyaningsih P., Volterra integral equations solved in fredholm form using walsh functions, ANZIAM J., 45, 2004, C269-C282.

[23] Sahu P. K. and Ray S. S., Legendre wavelets operational method for the numerical solution of nonlinear volterra integro-differential equations system, Appl. Math. Comp., 256, 2015, 715-723.

[24] Babolian E., Marzban H. R. and Salmani M., Using triangular orthogonal functions for solving Fredholm integral equations of the second kind, Appl. Math. Comp., 201, 2008, 452-464.

[25] Biazar J. and Asadi M. A., Galerkin RBF for Integro-differential Equations, British J. Math. Comp. Sci., 11(2), 2015, 1-9.

[26] Behzadi R., Tohidi E., Toutounian F., Numerical solution of weakly singular Fredholm integral equations via generalization of the Euler-Maclaurin summation formula, J. Taibah university for science, 8, 2014, 199-205.

[27] Ramane H. S., Shiralashetti S. C., Mundewadi R. A., Jummannaver R. B., Numerical solution of Fredholm Integral Equations Using Hosoya Polynomial of Path Graphs, American Journal of Numerical Analysis, 5(1), 2017, 11-15.