Summability Factor of Improper Integrals

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Abstract-In this paper, we have defined the summability for improper integrals and generalizing results of Ozgen and Mishra et al, we have established a theorem on indexed absolute Norlund summability factors of improper integral under sufficient conditions. Some auxiliary results (well-known) have also been deduced from the main result under suitable conditions. However, we established the main result on $\left|N, p, \delta; \mu\right|_k$ summability.

Keyword: Absolute summability, Norlund summability, Improper integrals

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1. LITERTURE SURVEY

Considering the $\left(N, p_n\right)$ and $\left(K,1,\alpha\right)$ summability, Parashar [12] obtained the minimum set of conditions for an infinite series to be $\left(K,1,\alpha\right)$ summable. In 1986, Bor [1] found a relationship between two summability techniques $\left(C,1\right)_k$ and $\left|\overline{N}, p_n\right|_k$ and in [2], he used the $\left|\overline{N}, p_n\right|_k$ for generalization of a theorem based on minimal set of sufficient conditions for infinite series. In 2016, Sonker and Munjal [12] determined a theorem on generalized absolute Cesaro summability with the sufficient conditions for infinite series and in [13] , they used the concept of triangular matrices for obtaining the minimal set of sufficient conditions of infinite series to be bounded. In 2017, Sonker and Munjal [13] found the approximation of the function $f \in Lip\left(\alpha, p\right)$ using infinite matrices of Cesaro submethod and in [14] ,they obtained boundness conditions of absolute summability factors. In this way by using the advanced summabuilitymethod,we can improve the quality of the filters. Borwein [3] extened many results on ordinary and absolute summability methods of integral.Canak [4] and Totur [15] worked on the concept of Cesaro summability with a very interesting result for integrals.In the same direction,weextened the results of Mazhar [6] with the help of some new generalized conditions and absolute Norlund summability $\left|N, p_n\right|_k$ factor for integrals.

2. INTRODUCTION

Let $\sum a_n$ be an infinite series with sequence of partial sums $\{s_n\}$. Let

$$\sigma_n = \frac{1}{n}\sum_{k=1}^{n} s_k \quad . \tag{2.1}$$

The series $\sum a_n$ is said to be $(C,1)$ summable, if

$$\lim_{n\to\infty}\sigma_n = s \tag{2.2}$$

where 's' is a finite number. The series $\sum a_n$ is said to be $\left|C,1\right|_k, k \geq 1,$ summable, if

$$\sum_{n=1}^{\infty} n^{k-1}\left|\sigma_n - \sigma_{n-1}\right|^k < \infty \tag{2.3}$$

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Let f be a real valued continuous function defined in the interval $[0,\infty)$ and $s(x) = \int_0^x f(t)\,dt$. We define the Cesaro mean of $s(x)$, denoted by $\tau(x)$, as

$$\tau(x) = \frac{1}{x}\int_0^x s(t)\,dt \tag{2.4}$$

$$\tau(x) = \frac{1}{x}\int_0^x (x-t) f(t)\,dt \quad . \tag{2.5}$$

The integral $\int_0^\infty f(t)\,dt$ is said to be summable $\left|C,1\right|$, if

$$\int_0^\infty \left|\tau'(x)\right| dx < \infty \tag{2.6}$$

and is said to be summable $|C,1|_k, k \geq 1$, if

$$\int_0^\infty x^{k-1} |\tau'(x)|^k dx < \infty . \qquad (2.7)$$

Let $p(x)$ be a real valued continuous function defined in the interval $[0,\infty)$ and $P(x) = \int_0^x p(t)dt$. We define the Norlund mean or (N, p) mean of $s(x)$ as a function $t(x)$ given by

$$t(x) = \frac{1}{P(x)} \int_0^x p(t)s(t)dt \ . \qquad (2.8)$$

The integral $\int_0^\infty f(t)dt$ is said to be summable $|N, p|$, if

$$\int_0^\infty |t'(x)| dx < \infty \ . \qquad (2.9)$$

It is said to be summable $|N, p|_k, k \geq 1$, if

$$\int_0^\infty \left(\frac{P(x)}{p(x)}\right)^{k-1} |t'(x)|^k dx < \infty . \qquad (2.10)$$

and is said to be summable $|N, p, \delta|_k, k \geq 1, \ \delta \geq 0$ and $\delta k \leq 1$, if

$$\int_0^\infty \left(\frac{P(x)}{p(x)}\right)^{\delta k+k-1} |t'(x)|^k dx < \infty . \qquad (2.11)$$

Further, for $\mu \geq 1$, the integral $\int_0^\infty f(t)dt$ is said to be summable $|N, p, \delta; \mu|_k$, if

$$\int_0^\infty \left(\frac{P(x)}{p(x)}\right)^{\mu(\delta k+k-1)} |t'(x)|^k dx < \infty . \qquad (2.12)$$

Clearly we have

$$s(x)-\tau(x)=\frac{1}{P(x)}\int_0^x P(t)\,f(t)\,dt\,.$$

Let

$$s(x)-\tau(x)=\nu(x) \qquad (2.13)$$

3. KNOWN RESULTS

Concerning absolute Cesaro summability $|C,1|_k$ *factors of integrals, Ozgen [8] obtained the following results :*

Theorem -3.1:

Let $\gamma(x)$ *be a positive monotonic non-decreasing function*

such that

$$\lambda(x)\gamma(x)=O(1),\ as\ x\to\infty \qquad (3.1.1)$$

$$\int_0^x u\left|\lambda''(u)\right|\gamma(u)\,du=O(1), \qquad (3.1.2)$$

$$\int_0^x \frac{|\nu(u)|^k}{u}\,du=O(\gamma(x)),\ as\ x\to\infty\ . \qquad (3.1.3)$$

Then the integral $\int_0^\infty f(t)dt$ *is summable* $|C,1|_k$, $k\geq 1$.

Recently, Mishra et al [7] extended Theorem-3.1.to $|C,1,\delta|_k$, $\delta\geq 0, \delta k\leq 1$ *summability by establishing the following theorem:*

Theorem-3.2.

Let $\chi(x)$ *be a positive non-decreasing function and there be two functions* $\beta(x)$ *and* $\varepsilon(x)$ *such that*

$$\left|\varepsilon'(x)\right|\leq\beta(x), \qquad (3.2.1)$$

$$\beta(x) \to 0, \text{ as } x \to \infty \qquad (3.2.2)$$

$$\int_0^\infty u\left|\beta'(u)\right|\chi(u)\,du < \infty, \qquad (3.2.3)$$

$$\left|\varepsilon(x)\right|\chi(x) = O(1), \qquad (3.2.4)$$

and

$$\int_0^x u^{\delta k-1}\left|v(u)\right|^k du = O(\chi(x)), \text{ as } x \to \infty. \quad (3.2.5)$$

Then the integrals $\int_0^\infty \varepsilon(t)\, f(t)dt$ *is summable* $\left|C,1,\delta\right|_k$, *for* $k \ge 1,\ \delta k \le 1.$

Very recently dealing with Norlund summability of Improper Integrals, Padhy et al [9]have established following result.

Theorem-3.3.

Let $p(0) > 0, p(x) \ge 0$ *and a non-increasing function.Let* $\chi(x)$ *be a positive non-decreasing function and there be two functions* $\beta(x)$ *and* $\varepsilon(x)$ *such that*

$$\left|\varepsilon'(x)\right| \le \beta(x), \qquad (3.3.1)$$

$$\beta(x) \to 0, \text{ as } x \to \infty \qquad (3.3.2)$$

$$\left|\varepsilon(x)\right|\chi(x) = O(1), \qquad (3.3.3)$$

$$\frac{1}{(p(x))^{k-1}}\int_0^x P(u)\left|\beta'(u)\right|\chi(u)\,du = O(1), \text{ as } x \to \infty \quad (3.3.4)$$

$$\frac{1}{(p(x))^{k-1}}\int_0^x P'(u)\left|\beta(u)\right|\chi(u)\,du = O(1), \text{ as } x \to \infty \quad (3.3.5)$$

and

$$\int_0^x \frac{p(t)\left|P'(t)\right|^k}{P(t)\left(p(t)\right)^k}\left|v(t)\right|^k dt = O\left(\chi(x)\right), \text{ as } x \to \infty \text{ .(3.3.6)}$$

Then the integral $\int_0^\infty \varepsilon(t) f(t) dt$ *is summable* $\left|N, p\right|_k$*, for* $k \geq 1$.

Extending the above result to $\left|N, p, \delta\right|_k, k \geq 1, \delta \geq 0 \text{ and } \delta k \leq 1$*, summability*

Paikray et al [10] have established the following theorem :

Theorem-3.4.

Let $p(0) > 0, p(x) \geq 0$ *and a non-increasing function. Further let* $\chi(x)$ *be a positive non-decreasing function and there be two functions* $\beta(x)$ *and* $\varepsilon(x)$ *such that*

$$\left|\varepsilon'(x)\right| \leq \beta(x), \text{ (3.4.1)}$$

$$\beta(x) \to 0, \text{ as } x \to \infty \text{ (3.4.2)}$$

$$\left|\varepsilon(x)\right| \chi(x) = O(1), \quad \text{(3.4.3)}$$

$$\left(\frac{P(x)}{p(x)}\right)^{\delta k} \frac{1}{\left(p(x)\right)^{k-1}} \int_0^x P(u)\left|\beta'(u)\right|\chi(u)\, du = O(1), \text{ as } x \to \infty \text{ (3.4.4)}$$

$$\left(\frac{P(x)}{p(x)}\right)^{\delta k} \frac{1}{\left(p(x)\right)^{k-1}} \int_0^x P'(u)\left|\beta(u)\right|\chi(u)\, du = O(1), \text{ as } x \to \infty \text{ (3.4.5)}$$

and

$$\int_0^x \left(\frac{P(t)}{p(t)}\right)^{\delta k-1} \left(\frac{P'(t)}{p(t)}\right)^k \left|v(t)\right|^k dt = O\left(\chi(x)\right), \text{ as } x \to \infty \text{ .(3.4.6)}$$

Then the integral $\int_0^\infty \varepsilon(t) f(t) dt$ *is summable* $\left|N, p, \delta\right|_k$*, for* $k \geq 1 \text{ and } 0 \leq \delta k \leq 1$.

4. MAIN RESULT

However, we establish the following result on $\left|N,p,\delta;\mu\right|_k$ summability. We prove

Theorem -4.1: *Let $\chi(x)$ be a positive non-decreasing function and there be two functions $\beta(x)$ and $\varepsilon(x)$ such that*

$$\left|\varepsilon'(x)\right| \le \beta(x), \quad (4.1.1) \qquad (4.1.1)$$

$$\beta(x) \to 0, \text{ as } x \to \infty \qquad (4.1.2)$$

$$\left|\varepsilon(x)\right|\chi(x) = O(1), \qquad (4.1.3)$$

$$\left(\frac{P(x)}{p(x)}\right)^{\mu(\delta k+k-1)-k+1} \frac{1}{\left(p(x)\right)^{k-1}} \int_0^x P(u)\left|\beta'(u)\right|\chi(u)\,du = O(1), \text{ as } x \to \infty \quad (4.1.4)$$

$$\left(\frac{P(x)}{p(x)}\right)^{\mu(\delta k+k-1)-k+1} \frac{1}{\left(p(x)\right)^{k-1}} \int_0^x P'(u)\left|\beta(u)\right|\chi(u)\,du = O(1), \text{ as } x \to \infty \quad (4.1.5)$$

and

$$\int_0^x \left(\frac{P(t)}{p(t)}\right)^{\mu(\delta k+k-1)-k} \left(\frac{P'(t)}{p(t)}\right)^k \left|v(t)\right|^k dt = O\left(\chi(x)\right), \text{ as } x \to \infty\,. \quad (4.1.6)$$

Then the integral $\int_0^\infty \varepsilon(t)\, f(t)dt$ *is summable* $\left|N,p,\delta,\mu\right|_k$, *for* $k \ge 1$ and $0 \le \delta k \le 1$.

Note :

The above theorem can be proved by using the concept of example that $\int_0^\infty x\left|\beta'(x)\right|\chi(x)\,dx < \infty$ is weaker $\int_0^\infty x\left|\varepsilon''(x)\right|\chi(x)\,dx < \infty$, and hence the introduction of the function $\{\beta(x)\}$ is justified.

Proof: It may be possible to choose the function $\beta(x)$ such that

$$\left|\varepsilon'(x)\right| \le \beta(x)$$

When $\varepsilon'(x)$ *oscillates,* $\beta(x)$ *may be chosen such that* $|\beta(x)|<|\varepsilon''(x)|$. *Hence* $\beta'(x)<|\varepsilon''(x)|$ *,so that*

$\int_0^\infty x|\beta'(x)|\chi(x)dx<\infty$ *is a weaker requirement that* $\int_0^\infty x|\varepsilon''(x)|\chi(x)dx<\infty$.

5. PROOF OF THE THEOREM

Let $T(x)$ be the (N,p_n) mean of the integral $\int_0^\infty \varepsilon(t)f(t)dt$ .The integral $\int_0^\infty \varepsilon(t)f(t)dt$ is $|N,p_n|_k$ summable, if

$$\int_0^x \left(\frac{P(x)}{p(x)}\right)^{\mu(\delta k+k-1)} |T'(t)|^k\, dt=O(1), \text{ as } x\to\infty, \tag{5.1}$$

where $T(x)$ is given by

$$T(x)=\frac{1}{P(x)}\int_0^x p(t)\left(\int_0^t \varepsilon(u)f(u)du\right)dt$$

$$=\frac{1}{P(x)}\int_0^x \varepsilon(u)f(u)du\int_u^x p(t)dt$$

$$=\frac{1}{P(x)}\int_0^x (P(x)-P(u))\varepsilon(u)f(u)du$$

$$=\int_0^x\left(1-\frac{P(u)}{P(x)}\right)\varepsilon(u)f(u)du \tag{5.2}$$

On differentiating both sides with respect to x ,we get

$$T'(x)=\frac{1}{(P(x))^2}\int_0^x P'(x)P(u)\varepsilon(u)f(u)du$$

$$=\frac{P'(x)\varepsilon(x)}{(P(x))^2}\int_0^x P(u)\,f(u)du-\frac{P'(x)}{(P(x))^2}\int_0^x \varepsilon'(u)\int_0^u P(t)\,f(t)dtdu$$

$$=\frac{P'(x)\varepsilon(x)v(x)}{P(x)}-\frac{P'(x)}{(P(x))^2}\int_0^x P(u)\varepsilon'(u)\left(\frac{1}{P(u)}\int_0^u P(t)\,f(t)dt\right)du$$

$$=\frac{P'(x)\varepsilon(x)v(x)}{P(x)}-\frac{P'(x)}{(P(x))^2}\int_0^x P(u)\varepsilon'(u)v(u)\,du$$

$$=T_1(x)+T_2(x). \qquad (5.3)$$

Applying Minkowski's inequality, we have

$$|T'(x)|^k=|T_1+T_2|^k<2^k\left(|T_1|^k+|T_2|^k\right) \ . \qquad (5.4)$$

Further, by Holder's inequality, we have

$$\int_0^x\left(\frac{P(t)}{p(t)}\right)^{\mu(\delta k+k-1)}|T_1(t)|^k\,dt=\int_0^x\left(\frac{P(t)}{p(t)}\right)^{\mu(\delta k+k-1)}\frac{|P'(t)|^k\,|v(t)|^k\,|\varepsilon(t)|^k}{|P(t)|^k}dt$$

$$=\int_0^x\left(\frac{P(t)}{p(t)}\right)^{\mu(\delta k+k-1)-k}\left(\frac{|P'(t)|}{p(t)}\right)^k|v(t)|^k\,|\varepsilon(t)|^{k-1}\,|\varepsilon(t)|dt$$

$$\leq\int_0^x\left(\frac{P(t)}{p(t)}\right)^{\mu(\delta k+k-1)-k}\left(\frac{|P'(t)|}{p(t)}\right)^k|v(t)|^k\,|\varepsilon(t)|dt$$

$$=|\varepsilon(x)|\int_0^x\left(\frac{P(t)}{p(t)}\right)^{\mu(\delta k+k-1)-k}\left(\frac{|P'(t)|}{p(t)}\right)^k|v(t)|^k\,dt-\int_0^x\left|\varepsilon'(t)\left(\int_0^t\left(\frac{P(y)}{p(y)}\right)^{\mu(\delta k+k-1)-k}\left(\frac{|P'(t)|}{p(y)}\right)^k|v(y)|^k\,dy\right)\right|dt$$

$$=O(1)|\varepsilon(x)|\chi(x)-\int_0^x\beta(t)\chi(t)\,dt$$

$$=O(1)-\beta(x)\int_0^x \chi(u)\,du+\int_0^x |\beta'(t)|\left(\int_0^t \chi(u)\,du\right)dx$$

$$\leq O(1)-\beta(x)\int_0^x \chi(u)\,du+\int_0^x t|\beta'(t)|\chi(t)\,dt$$

$$=O(1),\ as\ x\to\infty. \tag{5.5}$$

Next

$$\int_0^x \left(\frac{P(t)}{p(t)}\right)^{(\mu(\delta k+k-1))} |T_2(t)|^k\,dt=\int_0^x \left(\frac{P(t)}{p(t)}\right)^{\mu(\delta k+k-1)} \left|\frac{P'(t)}{(P(t))^2}\int_0^t P(u)\varepsilon'(u)v(u)\,du\right|^k dt$$

$$\leq \int_0^x \left(\frac{P(t)}{p(t)}\right)^{\mu(\delta k+k-1)-k+1} \frac{P'(t)}{(p(t))^{k-1}(P(t))^2}\left(\int_0^t (P(u))^k|\varepsilon'(u)|^k|v(u)|^k\,du\right)\left(\frac{1}{P(t)}\int_0^t P''(u)\,du\right)^{k-1} dt$$

$$=\int_0^x |P(u)\varepsilon'(u)|^{k-1}|P(u)\varepsilon'(u)||v(u)|^k\,du\int_u^x \left(\frac{P(t)}{p(t)}\right)^{\mu(\delta k+k-1)-k+1}\frac{P'(t)}{(p(t))^{k-1}(P(t))^2}\,dt$$

$$=O\left(\left(\frac{P(x)}{p(x)}\right)^{\mu(\delta k+k-1)-k+1}\frac{1}{(p(x))^{k-1}}\right)\int_0^x |P(u)\varepsilon'(u)||v(u)|^k\left(\frac{1}{P(u)}-\frac{1}{P(x)}\right)du$$

$$\leq O\left(\left(\frac{P(x)}{p(x)}\right)^{\mu(\delta k+k-1)-k+1}\frac{1}{(p(x))^{k-1}}\right)\int_0^x |P(u)\varepsilon'(u)||v(u)|^k\left(\frac{1}{P(u)}\right)du$$

$$=O\left(\left(\frac{P(x)}{p(x)}\right)^{\mu(\delta k+k-1)-k+1}\frac{1}{(p(x))^{k-1}}\right)P(x)|\varepsilon'(x)|\int_0^x |v(u)|^k\left(\frac{1}{P(u)}\right)du-O\left(\left(\frac{P(x)}{p(x)}\right)^{\mu(\delta k+k-1)-k+1}\frac{1}{(p(x))^{k-1}}\right)\int_0^x \left(P(u)|\varepsilon'(u)|\right)'\int_0^u |v(t)|^k\left(\frac{1}{P(t)}\right)dt\,du$$

$$=O\left(\left(\frac{P(x)}{p(x)}\right)^{\mu(\delta k+k-1)-k+1}\frac{1}{(p(x))^{k-1}}\right)\left\{P(x)|\beta(x)|\chi(x)-\int_0^x \left(P(u)|\beta(u)|\right)'\chi(u)\,du\right\}$$

$$=O\left(\left(\frac{P(x)}{p(x)}\right)^{\mu(\delta k+k-1)-k+1}\frac{1}{\left(p(x)\right)^{k-1}}\right)\left\{P(x)\left|\beta(x)\right|\chi(x)-\int_0^x P'(u)\left|\beta(u)\right|\chi(u)du-\int_0^x\left(P(u)\beta'(u)\right)\chi(u)du\right\}$$

$$=O(1), \quad \text{as} \quad x\to\infty . \qquad (5.6)$$

On collecting (5.2)-(5.6), we have

$$\int_0^x\left(\frac{P(t)}{p(t)}\right)^{\mu(\delta k+k-1)}\left|T'(t)\right|^k dt=O(1) \quad , \quad \text{as} \quad x\to\infty .$$

This completes the proof of the theorem.

6. CONCLUSION

The main result of this research article is an attempt to formulate the problem of absolute summability factor of integrals which make a more modified filter. Through the investigation, we concluded that the improper integral is absolute Norlund summable under the minimal sufficient conditions. Further, this study has a number of direct applications in rectification of signals in FIR filter (finite impulse response filter) and IIR filter (infinite impulse response filter). In a nut shell absolute summability method is a motivation for the researchers, interested in studies of improper integrals.

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