Global Exponential Stability of Impulsive Delay Differential Equations Using Lyapunov- Razumikhin Technique

[a]Sanjay K. Srivastava, [b]Neeti Bhandari*,
[a]Department of Applied Sciences, Beant College of Engineering and Technology, Gurdaspur – 143521, Punjab, India.
[b] I.K. Gujral Punjab Technical University, Jalandhar, Punjab, India.
∗ Corresponding author email Id: neetiumat@gmail.com

Abstract

This paper studies the global exponential stability of the impulsive delay differential by using Lyapunov functions and Razumikhin techniques. The obtained results extend and generalize some results existing in the literature. Moreover, the Razumikhin condition obtained is very simple.

Keywords: Impulsive delay differential equations, Global exponential stability, Razumikhin techniques, Lyapunov functions.

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1. INTRODUCTION

There has been a growing interest in the theory of impulsive differential equations in recent years since they provide a natural framework for mathematical modelling of many real world phenomena. Impulsive differential equations have attracted many researchers attention due to their wide application in many fields such as control technology, drug administration and threshold theory in biology and the like. Such systems arise naturally from a variety of applications such as orbital transfer of satellites, impact and constrained mechanics, sampled data systems, inspection processes in operations research and ecosystem management. There are several research works which appeared in the literature on the stability of impulsive differential equations.

In, recent years stability of impulsive delay differential equations has been extensively studied. See [1-4,6-11] and the references therein. Recently there have been some research work in this direction in [11-15], the author investigated the uniform asymptotic stability and global exponential stability of impulsive delay differential equation. In [10], the author obtained some results on exponential stabilization of impulsive delay differential equations. The method of Lyapunov functions and Razumikhin technique have been widely applied to stability analysis of various impulsive delay differential equations and they have also proved to be powerful tool in the investigation of impulsive delay differential equations [6,8,10]. The aim of this work is to establish global exponential stability criteria for impulsive delay systems by employing the Razumikhin technique which illustrate that impulses do contribute to the stabilization of some delay differential systems. Our results show that impulses may be used as a control to stabilize the underlying continuous system.

2.PRELIMINARIES

Let R denote the set of real numbers, R_+ the set of nonnegative real numbers and R^n the n-dimensional real space equipped with the Euclidean norm $\|\cdot\|$. Let N denote the set of positive integers, i.e.,$N = \{1,2,\dots\}$. Given a constant $\tau > 0$, we equip the linear space $PC([-\tau, 0], R^n)$ with the norm $\|\cdot\|_\tau$ defined by $\|\psi\|_\tau = \sup_{-\tau \le s \le 0} \|\psi(s)\|$.

Consider the following impulsive delay system:

$$\begin{cases} x'(t) = F(t, x_t), & t \ne t_k \\ \Delta x(t_k) = I_k\left(t_k, x_{t_k^-}\right), & k \in N, \\ x_{t_0} = \phi, & \end{cases} \tag{2.1}$$

Where $F, I_k : R_+ \times PC([-\tau, 0], R^n) \to R^n$; $\phi \in PC([-\tau, 0], R^n)$; $0 \le t_0 < t_1 < t_2 < \dots < t_k < \dots$, with $t_k \to \infty$ as $k \to \infty$; $\Delta x(t) = x(t) - x(t^-)$; and $x_t, x_{t^-} \in PC([-\tau, 0], R^n)$ are defined by $x_t(s) = x(t+s), x_{t^-}(s) = x(t^- + s)$ for $-\tau \le s \le 0$, respectively.We shall assume that $F(t, 0) = I_k(t, 0) = 0$ for all $t \in R_+$ and $k \in N$ so that system (2.1) have the trivial solution. Given a constant $\tau > 0$, we equip the linear space $PC([-\tau, 0], R^n)$ with the norm $\|\cdot\|_\tau = \sup_{-\tau \le s \le 0} \|\psi(s)\|$. Denote $x_t = x(t, t_0, \phi)$ the solution of (2.1) such that $x_{t_0} = \phi$. We further assume that all the solution $x(t)$ of (2.1) are continuous except at t_k, $k \in N$, at which $x(t)$ is right continuous, i.e., $x_{t_k^+} = x(t_k)$, $k \in N$.

Definition 2.1:
Function $V : R_+ \times R^n \to R_+$ is said to belong to the class v_0 if
(i) V is continuous in each of the sets $[t_{k-1}, t_k) \times R^n$ and for each $x \in R^n, t \in [t_{k-1}, t_k)$, And $k \in N$, $lim_{(t,y)\to(t_k^-, x)} V(t, y) = V(t_k^-, x)$ exists; and

(ii) $V(t, x)$ is locally Lipschitzian in all $x \in R^n$, and for all $t \ge t_0$, $V(t, 0) \equiv 0$.

Definition 2.2. Given a function $V : R_+ \times R^n \to R_+$, the upper right-hand derivative of V with respect to system (2.1) is defined by

$$D^+V(t, \psi(0)) = \lim_{h \to 0+} sup \frac{1}{h} \{V(t + h, \psi(0) + h\,F(t, \psi)) - V(t, \psi(0))\}$$

for $(t, \psi) \in R_+ \times PC([-\tau, 0], R^n)$.

Definition 2.3. The trivial solution of (2.1) is said to be globally exponentially stable if there exist some constants $\alpha > 0$ and $M \ge 1$ such that for any initial data $x_{t_0} = \phi$

$$\|x(t, t_0, \phi)\| \le M\|\phi\|_\tau\, e^{-\alpha(t-t_0)}, \quad t \ge t_0,$$

where $(t_0, \phi) \in R_+ \times PC([-\tau, 0], R^n)$.

3.The Lyapunov-Razumikhin Method

In this section, we shall present some Razumikhin-type theorems on global exponential stability for system (2.1) based on the Lyapunov–Razumikhin method. Our results show that impulses play an important role in stabilizing delay differential systems.

Theorem 3.1. Assume that there exist a function $V \in v_0$ and several positive constants p, c, c_1, c_2, $\lambda > 0, \gamma \geq 1, \sigma > 1$ and $\sigma - \lambda \geq$ c such that

(i) $c_1\|x\|^p \leq V(t,x) \leq c_2\|x\|^p$, for any $t \in R_+$ *and* $x \in R^n$;

(ii) $D^+V(t,\varphi(0)) \leq cV(t,\varphi(0))$, for all $t \in [t_{k-1}, t_k), k \in N$, whenever $qV(t,\varphi(0)) \geq V(t+s,\varphi(s)$ for $s \in [-\tau, 0]$, where $q \geq \gamma e^{\lambda\tau}$ is a constant;

(iii) $V(t_k, \varphi(0) + I_k(t_k,\varphi)) \leq d_k V(t_k^-, \varphi(0))$, where $0 < d_{k-1} \leq 1, \forall\, k \in N$ are constants;

(iv) $\gamma \geq 1/d_{k-1}$ and $In\, d_{k-1} < -\lambda\sigma(t_k - t_{k-1}), k \in N$.

Then the zero solution of the impulsive retarded differential equation (2.1) is globally exponentially stable with convergence rate $\frac{\lambda}{p}$ for any time delays $\tau \in (0,\infty)$.

Proof: Let $x(t) = x(t, t_0, \phi)$ be any solution of the impulsive system (2.1) with the initial condition $x_{t_0} = \phi$, and $v(t) = V(t,x)$.

We shall show that

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \quad t \in [t_{k-1}, t_k),\ k \in N. \tag{3.1}$$

Let $\gamma \geq sup_{k\in N}\left\{\frac{1}{d_{k-1}}\right\}$. From condition (iv), we can choose a positive constant $M > 0$ such that

$$0 < c_2 e^{\lambda\sigma(t_1-t_0)} \leq M \leq c_2\, \gamma e^{\lambda\tau - \lambda\sigma(t_1-t_0)} e^{\lambda\sigma(t_1-t_0)} \tag{3.2}$$

It then follows that

$$0 < c_2\|\phi\|_\tau^p < c_2\|\phi\|_\tau^p e^{\sigma(t_1-t_0)} \leq M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)} \tag{3.3}$$

We first prove that

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \quad t \in [t_0, t_1). \tag{3.4}$$

To do this, we only need to prove that

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)}, \quad t \in [t_0, t_1). \tag{3.5}$$

If (3.5) is not true, then by (3.3) there exists $\bar{t} \in (t_0, t_1)$ such that

$$v(\bar{t}) > M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)} \geq c_2\|\phi\|_\tau^p e^{\sigma(t_1-t_0)} > c_2\|\phi\|_\tau^p \geq v(t_0+s), \quad s \in [-\tau, 0],$$

Which implies that there exists $t^* \in (t_0, \bar{t})$ such that

$$v(t^*) = M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)} \text{ and } v(t) \leq v(t^*),\ t \in [t_0-\tau, t^*], \tag{3.6}$$

and there exists $t^{**} \in [t_0, t^*)$ such that

$$v(t^{**}) = c_2\|\phi\|_\tau^p \text{ and } v(t^{**}) \leq v(t) \leq v(t^*),\ t \in [t^{**}, t^*]. \tag{3.7}$$

Hence, for any $s \in [-\tau, 0]$, by (3.2) and (3.7), we get

$$\begin{aligned} v(t+s) &\leq M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)} \leq c_2\, \gamma e^{\lambda\tau-\lambda\sigma(t_1-t_0)} e^{\lambda\sigma(t_1-t_0)} \|\phi\|_\tau^p e^{-\lambda(t_1-t_0)} \leq \gamma e^{\lambda\tau} c_2\|\phi\|_\tau^p \\ &= \gamma e^{\lambda\tau}\, v(t^{**}) \leq qv(t^{**}) \leq qv(t), \quad t \in [t^{**}, t^*] \end{aligned} \tag{3.8}$$

and thus by (3.8) and condition (ii), for $t \in [t^{**}, t^*]$, we get $D^+(v(t)) \leq cv(t) \leq \lambda\sigma\, v(t)$. It follows from (3.2),(3.6) and (3.7) that

$$\begin{aligned} v(t^*) &\leq v(t^{**}) e^{\lambda\sigma(t^*-t^{**})} < c_2\|\phi\|_\tau^p e^{\lambda\sigma(t_1-t_0)} < c_2\|\phi\|_\tau^p e^{\sigma(t_1-t_0)} \\ &\leq M\|\phi\|_\tau^p e^{-\lambda(t_1-t_0)} = v(t^*) \end{aligned}$$

Which is a contradiction. Hence (3.4) holds and then (3.1) is true for $k = 1$.
Now we assume that (3.1) holds for $k = 1,2,\dots,m (m \in N, m \geq 1)$, i.e.,

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \quad t \in [t_{k-1}, t_k), k = 1,2,\dots,m. \tag{3.9}$$

Next, we shall show that (3.1) holds for $k = m+1$, i.e.

$$v(t) \leq M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}, \quad t \in [t_m, t_{m+1}) \tag{3.10}$$

Suppose (3.10) is not true. Then we define $\bar{t} = \inf\{t \in [t_m, t_{m+1}) \big| v(t) > M\|\phi\|_\tau^p e^{-\lambda(t-t_0)}\}$.
From conditions (iii), (iv) and (3.9), we get

$$\begin{aligned} v(t_m) &\leq d_m M\|\phi\|_\tau^p e^{-\lambda(t_m-t_0)} = d_m M\|\phi\|_\tau^p e^{\lambda(\bar{t}-t_m)} e^{-\lambda(\bar{t}-t_0)} < d_m e^{\lambda(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} \\ &< e^{-\lambda\sigma(t_{m+1}-t_m)} e^{\lambda(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} \\ &< M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} \end{aligned} \tag{3.11}$$

and hence $\bar{t} \neq t_m$. From the continuity of $v(t)$ on the interval $[t_m, t_{m+1})$, we have

$$v(\bar{t}) = M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)}, \quad \text{and} \quad v(t) \leq v(\bar{t}), \quad t \in [t_m, \bar{t}]. \tag{3.12}$$

From (3.11), we know that there exists $t^* \in (t_m, \bar{t})$ such that

$$v(t^*) = d_m e^{\lambda(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)}, \quad \text{and} \quad v(t^*) \leq v(t) \leq v(\bar{t}), \ t \in [t^*, \bar{t}]. \tag{3.13}$$

On the other hand, for $t \in [t^*, \bar{t}]$ and $s \in [-\tau, 0]$, either $t + s \in [t_0 - \tau, t_m)$ or $t + s \in [t_m, \bar{t}]$.
If $t + s \in [t_0 - \tau, t_m)$ from (3.9), we obtain

$$\begin{aligned} v(t+s) &\leq M\|\phi\|_\tau^p e^{-\lambda(t+s-t_0)} = M\|\phi\|_\tau^p e^{-\lambda(t-t_0)} e^{-\lambda s} \\ &\leq M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} e^{\lambda(\bar{t}-t)} e^{\lambda\tau} \\ &\leq e^{\lambda\tau} e^{\lambda(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} \end{aligned} \tag{3.14}$$

while, if $t + s \in [t_m, \bar{t}]$, from (3.12), then

$$v(t+s) \leq M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} \leq e^{\lambda\tau} e^{\lambda(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} \tag{3.15}$$

In any case however, (3.13)-(3.15) imply that, for any $s \in [-\tau, 0]$, we have

$$v(t+s) \leq e^{\lambda\tau} e^{\lambda(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} \leq \gamma e^{\lambda\tau} v(t) \leq q v(t), \quad t \in [t^*, \bar{t}]. \tag{3.16}$$

Finally, by (3.16) and condition (ii), we have $D^+(v(t)) \leq \lambda\sigma\, v(t)$. Thus, in view of condition (iv), we have

$$\begin{aligned} v(\bar{t}) &\leq v(t^*) e^{\lambda\sigma(\bar{t}-t^*)} = d_m e^{\lambda(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} e^{\lambda\sigma(\bar{t}-t^*)} \\ &< e^{-\lambda\sigma(t_{m+1}-t_m)} e^{\lambda(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} e^{\lambda\sigma(\bar{t}-t^*)} \\ &< M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} e^{\lambda\sigma(\bar{t}-t_0)} \\ &< M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} = v(\bar{t}) \end{aligned}$$

Which is a contradiction. This implies the assumption is not true, and hence (3.10) holds. Therefore, by some mathematical induction, we obtain (3.1) holds for any $k \in N$. Then from condition (i), we have

$$\|x\| \leq M^*\|\phi\|_\tau e^{-\frac{\lambda}{p}(t-t_0)}, \quad t \in [t_{k-1}, t_k), k \in N$$

where $M^* \geq max\left\{1, [\frac{M}{c_1}]^{\frac{1}{p}}\right\}$, which implies that the zero solution of the impulsive system (2.1) is globally exponentially stable with convergence rate $\frac{\lambda}{p}$. The proof is completed.

Theorem 3.2. Assume that there exist a function $V \in v_0$ and several positive constants p, c, c_1, $c_2, \sigma, \lambda > 0, \gamma \geq 1$, and $\sigma - \lambda \geq c$ such that

(i) $c_1\|x\|^p \leq V(t,x) \leq c_2\|x\|^p$, for any $t \in R_+$ and $x \in R^n$;

(ii) $D^+V(t,\varphi(0)) \leq cV(t,\varphi(0))$, for all $t \in [t_{k-1}, t_k), k \in N$, whenever $qV(t,\varphi(0)) \geq V(t+s,\varphi(s))$ for $s \in [-\tau, 0]$, where $q \geq \gamma e^{\lambda\tau}$ is a constant;

(iii) $V(t_k, \varphi(0) + I_k(t_k, \varphi)) \leq d_k V\big(t_k^-, \varphi(0)\big)$, where $0 < d_{k-1} \leq 1, \forall\, k \in N$ are constants;

(iv) $\gamma \geq 1/d_{k-1}$ and In $In\, d_{k-1} < -(\sigma + \lambda + c)(t_k - t_{k-1}), k \in N$.

Then the zero solution of the impulsive retarded differential equation (2.1) is globally exponentially stable with convergence rate $\frac{\lambda}{p}$ for any time delays $\tau \in (0, \infty)$.

Proof: Let $x(t) = x(t, t_0, \phi)$ be any solution of the impulsive system (2.1) with the initial condition $x_{t_0} = \phi$, and $v(t) = V(t, x)$.

We shall show that

$$v(t) \leq M\|\phi\|_\tau^p e^{-(\lambda+c)(t-t_0)},\ t \in [t_{k-1}, t_k),\ k \in N. \tag{3.17}$$

Let $\gamma \geq sup_{k\in N}\left\{\frac{1}{d_{k-1}}\right\}$. From condition (iv), we can choose a positive constant $M > 0$ such that

$$0 < \ c_2 e^{(\sigma+\lambda+c)(t_1-t_0)} \leq M \leq \ c_2 \gamma e^{(\lambda+c)\tau-(\sigma+\lambda+c)(t_1-t_0)} e^{(\sigma+\lambda+c)(t_1-t_0)} \tag{3.18}$$

It then follows that

$$0 < \mathrm{c}_2\|\phi\|_\tau^p < \ \mathrm{c}_2\|\phi\|_\tau^p e^{\sigma(t_1-t_0)} \leq M\|\phi\|_\tau^p e^{-(\lambda+c)(t_1-t_0)} \tag{3.19}$$

We first prove that

$$v(t) \leq M\|\phi\|_\tau^p e^{-(\lambda+c)(t-t_0)}, t \in [t_0, t_1). \tag{3.20}$$

To do this, we only need to prove that

$$v(t) \leq M\|\phi\|_\tau^p e^{-(\lambda+c)(t_1-t_0)}, t \in [t_0, t_1). \tag{3.21}$$

If (6) is not true, then by (3.19) there exists $\bar{t} \in (t_0, t_1)$ such that

$$v(\bar{t}) > \ M\|\phi\|_\tau^p e^{-(\lambda+c)(t_1-t_0)} \geq \mathrm{c}_2\|\phi\|_\tau^p e^{\sigma(t_1-t_0)} > \mathrm{c}_2\|\phi\|_\tau^p \geq v(t_0 + s), \quad s \in [-\tau, 0],$$

Which, implies that there exists $\mathrm{t}^* \in (t_0, \bar{t})$ such that

$$v(\mathrm{t}^*) = \ M\|\phi\|_\tau^p e^{-(\lambda+c)(t_1-t_0)} \text{ and } v(t) \leq v(\mathrm{t}^*),\ \ \mathrm{t} \in [t_0 - \tau, \mathrm{t}^*], \tag{3.22}$$

and there exists $\mathrm{t}^{**} \in [\mathrm{t}_0, \mathrm{t}^*)$ such that

$$v(\mathrm{t}^{**}) = \mathrm{c}_2\|\phi\|_\tau^p \ \text{ and } \ v(\mathrm{t}^{**}) \leq v(t) \leq v(\mathrm{t}^*),\ \ t \in [\mathrm{t}^{**}, \mathrm{t}^*]. \tag{3.23}$$

Hence, for any $s \in [-\tau, 0]$, by (3.18) and (3.23), we get

$$\begin{aligned} v(t+s) &\leq M\|\phi\|_\tau^p e^{-(\lambda+c)(t_1-t_0)} \leq c_2\ \gamma e^{(\lambda+c)\tau-(\sigma+\lambda+c)(t_1-t_0)} e^{(\sigma+\lambda+c)(t_1-t_0)} \|\phi\|_\tau^p e^{-(\lambda+c)(t_1-t_0)} \\ &\leq \ \gamma e^{(\lambda+c)\tau} \mathrm{c}_2\|\phi\|_\tau^p = \gamma e^{(\lambda+c)\tau}\ v(\mathrm{t}^{**}) \leq q v(\mathrm{t}^{**}) \leq q v(t), t \in [\mathrm{t}^{**}, \mathrm{t}^*] \end{aligned} \tag{3.24}$$

and thus by (3.24) and condition(ii), for $t \in [\mathrm{t}^{**}, \mathrm{t}^*]$, we get $D^+\big(v(t)\big) \leq cv(t) \leq (\sigma - \lambda - c)\ v(t)$. It follows from (3.18),(3.22) and (3.23) that

$$\begin{aligned} v(\mathrm{t}^*) &\leq v(\mathrm{t}^{**})\mathrm{e}^{(\sigma-\lambda-c)(\mathrm{t}^*-\mathrm{t}^{**})} < \mathrm{c}_2\|\phi\|_\tau^p e^{(\sigma-\lambda-c)(t_1-t_0)} < \mathrm{c}_2\|\phi\|_\tau^p e^{\sigma(t_1-t_0)} \\ &= \mathrm{c}_2\|\phi\|_\tau^p e^{(\sigma+\lambda+c)(t_1-t_0)} e^{(-\lambda-c)(t_1-t_0)} \\ &\leq M\|\phi\|_\tau^p e^{(-\lambda-c)(t_1-t_0)} = v(\mathrm{t}^*) \end{aligned}$$

Which, is a contradiction. Hence (3.20) holds and then (3.17) is true for $k = 1$.

Now we assume that (3.17) holds for $k = 1,2, \dots, m (m \in N, m \geq 1)$, i.e.,

$$v(t) \leq M\|\phi\|_\tau^p e^{(-\lambda-c)(t-t_0)}, t \in [t_{k-1}, t_k), k = 1,2, \dots, m. \tag{3.25}$$

Next, we shall show that (3.17) holds for $k = m + 1$, i.e.

$$v(t) \leq M\|\phi\|_\tau^p e^{(-\lambda-c)(t-t_0)}, t \in [t_m, t_{m+1}) \tag{3.26}$$

Suppose (3.26) is not true. Then we define $\bar{t} = \inf\left\{t \in [t_m, t_{m+1}) \middle| v(t) > M\|\phi\|_\tau^p e^{(-\lambda-c)(t-t_0)}\right\}$.

From conditions (iii), (iv) and (3.25), we get

$$\begin{aligned} v(t_m) &\leq d_m M\|\phi\|_\tau^p e^{(-\lambda-c)(t_m-t_0)} \\ &= d_m M\|\phi\|_\tau^p e^{(\lambda+c)(\bar{t}-t_m)} e^{-(\lambda+c)(\bar{t}-t_0)} \\ &< d_m e^{(\lambda+c)(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-(\lambda+c)(\bar{t}-t_0)} \\ &< e^{-(\sigma+\lambda+c)(t_{m+1}-t_m)} e^{(\lambda+c)(t_{m+1}-t_m)}\ M\|\phi\|_\tau^p e^{-(\lambda+c)(\bar{t}-t_0)} \end{aligned}$$

$$< M\|\phi\|_\tau^p e^{-(\lambda+c)(\bar{t}-t_0)} \tag{3.27}$$

and hence $\bar{t} \neq t_m$. From the continuity of $v(t)$ on the interval $[t_m, t_{m+1})$, we have

$$v(\bar{t}) = M\|\phi\|_\tau^p e^{-(\lambda+c)(\bar{t}-t_0)}, \quad \text{and} \quad v(t) \le v(\bar{t}), \quad t \in [t_m, \bar{t}]. \tag{3.28}$$

From (3.27), we know that there exists $t^* \in (t_m, \bar{t})$ such that

$$v(t^*) = d_m e^{(\lambda+c)(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-(\lambda+c)(\bar{t}-t_0)}, \text{ and } v(t^*) \le v(t) \le v(\bar{t}), \ t \in [t^*, \bar{t}]. \tag{3.29}$$

On the other hand, for $t \in [t^*, \bar{t}]$ and $s \in [-\tau, 0]$, either $t+s \in [t_0-\tau, t_m)$ or $t+s \in [t_m, \bar{t}]$. If $t+s \in [t_0-\tau, t_m)$ from (3.25), we obtain

$$\begin{aligned} v(t+s) &\le M\|\phi\|_\tau^p e^{-(\lambda+c)(t-t_0)} e^{-(\lambda+c)s} \\ &\le M\|\phi\|_\tau^p e^{-(\lambda+c)(\bar{t}-t_0)} e^{(\lambda+c)(\bar{t}-t)} e^{(\lambda+c)\tau} \\ &\le e^{(\lambda+c)\tau} e^{(\lambda+c)(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-(\lambda+c)(\bar{t}-t_0)} \end{aligned} \tag{3.30}$$

While, if $t+s \in [t_m, \bar{t}]$, from (3.28), then

$$v(t+s) \le M\|\phi\|_\tau^p e^{-(\lambda+s)(\bar{t}-t_0)} \le e^{(\lambda+c)\tau} e^{\lambda(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-(\lambda+c)(\bar{t}-t_0)} \tag{3.31}$$

In any case however, (3.29)-(3.31) imply that, for any $s \in [-\tau, 0]$, we have

$$\begin{aligned} v(t+s) &\le e^{(\lambda+c)\tau} e^{(\lambda+c)(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-(\lambda+c)(\bar{t}-t_0)} \le \gamma e^{(\lambda+c)\tau} v(t^*) \\ &\le \gamma e^{(\lambda+c)\tau} v(t) \le q v(t), t \in [t^*, \bar{t}] \end{aligned} \tag{3.32}$$

Finally, by (3.32) and condition (ii), we have $D^+(v(t)) \le (\sigma - \lambda - c)\, v(t)$. Thus, in view of condition (iv), we have

$$\begin{aligned} v(\bar{t}) &\le v(t^*) e^{(\sigma-\lambda-c)(\bar{t}-t^*)} \\ &= d_m e^{(\lambda+c)(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-(\lambda+c)(\bar{t}-t_0)} e^{(\sigma-\lambda-c)(\bar{t}-t^*)} \\ &< e^{-(\sigma+\lambda+c)(t_{m+1}-t_m)} e^{(\lambda+c)(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-(\lambda+c)(\bar{t}-t_0)} e^{(\sigma-\lambda-c)(\bar{t}-t^*)} \\ &< e^{-(\sigma+\lambda+c)(t_{m+1}-t_m)} e^{(\lambda+c)(t_{m+1}-t_m)} M\|\phi\|_\tau^p e^{-(\lambda+c)(\bar{t}-t_0)} e^{(\sigma-\lambda-c)(\bar{t}-t^*)} \\ &< M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} e^{\lambda\sigma(\bar{t}-t_0)} \\ &< M\|\phi\|_\tau^p e^{-\lambda(\bar{t}-t_0)} = v(\bar{t}) \end{aligned}$$

Which is a contradiction. This implies the assumption is not true, and hence (3.26) holds. Therefore, by some mathematical induction, we obtain (3.17) holds for any $k \in N$. Then from condition (i), we have

$$\|x\| \le M^*\|\phi\|_\tau e^{-\frac{\lambda}{p}(t-t_0)}, \quad t \in [t_{k-1}, t_k), k \in N$$

where $M^* \ge max\left\{1, [\frac{M}{c_1}]^{\frac{1}{p}}\right\}$, which implies that the zero solution of the impulsive system (2.1) is globally exponentially stable with convergence rate $\frac{\lambda}{p}$. The proof is completed.

We can see from Theorem 3.1 and 3.2, impulses have played an important role in exponentially stabilizing a delay differential system.

Next, we apply te previous theorems to the following linear impulsive delay differential system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau(t)), \ t \neq t_k, t \ge t_0 \\ \Delta x(t) = C_k x(t^-), \ t = t_k, \ k \in N \\ x_{t_0} = \phi, \end{cases} \tag{3.33}$$

Corollary 3.1 Suppose there exists some constants $\sigma, \lambda > 0$, and $\gamma \ge 1$ such that

(i) for some constant $q \ge \gamma e^{\lambda\tau}, \lambda_{max}(A^T + A + E) + q\|B\|^2 \le \sigma - \lambda$

(ii) $\gamma \ge \frac{1}{\|E+C_{k-1}\|}$ and $In\|E + C_{k-1}\| < -(\sigma+\lambda)(t_k - t_{k-1})$, where $C_0 = E, k \in N$.

Then system (3.33) is globally exponentially stable and its convergence rate is $\lambda/2$.

Proof. It follows from above theorem by selecting Lyapunov function $V(x) = \|x\|^2$.

Example 3.1 Consider the following linear impulsive retarded dynamical system.

$$\begin{cases} \dot{x}(t) = Ax(t) + Bx\left(t - \frac{2}{5}(1 + e^{-t})\right), \quad t \neq t_k, t \geq t_0 \\ \Delta x(t) = C_k x(t^-), \quad t = t_k, k \in N \\ x_{t_0} = \phi, \end{cases} \tag{3.34}$$

where

$$A = \begin{bmatrix} 0.1 & 0.2 & -0.1 \\ 0.2 & 0.15 & 0.3 \\ 0 & 0.24 & 0.1 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.12 & 0.02 & 0 \\ 0.12 & -0.2 & 0.05 \\ 0 & 0.14 & -0.1 \end{bmatrix}$$

and

$$C_k = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & -0.8 & 0 \\ 0 & 0 & -0.4 \end{bmatrix}$$

It is easy to check that for the time delay $\tau = 0.8$, the corresponding system without impulses is unstable. The numerical simulation of this retarded dynamical system with respect to initial functions;

$$\phi_1(t) = \begin{cases} 0, & t \epsilon[-0.8,0) \\ 2.8, & t = 0; \end{cases} \qquad \phi_2(t) = \begin{cases} 0, & t \epsilon[-0.8,0) \\ -1.4, & t = 0; \end{cases} \qquad \phi_3(t) = \begin{cases} 0, & t \epsilon[-0.8,0) \\ 2.1, & t = 0; \end{cases}$$

is given in Fig. 1.

It is easy to see that $\lambda_{max}(A^T + A + E) = 1.8819, \|B\|^2 = \lambda_{max}(BB^T) = 0.0844$ and $\|E + C_k\| = 0.6$. By taking $q = 25, \gamma = 5, \lambda = 2, \sigma = 6.2$ and $\mathrm{t_{k+1}} - \mathrm{t_k} = 0.06$, it is easy to verify that all the conditions of Corollary 3.1 hold:

(i) $q = 25 \geq \gamma e^{\lambda\tau} = 24.7652, \lambda_{max}(A^T + A + E) + q\|B\|^2 = 3.9919 \leq \sigma - \lambda = 4.2$;

(ii) $In\ \|E + C_k\| = -0.5108 < -(\sigma + \lambda)(\mathrm{t_{k+1}} - \mathrm{t_k}) = -0.4920$;

Which, means the impulsive retarded dynamical system (3.34) is globally exponentially stable with convergence rate 1. This conclusion cannot be delivered by applying the corresponding exponential stability results for impulsive retarded differential equations given in the literature [1,2], since the length of the impulsive integrals is excessively less than the time delays, i.e.,$t_k - t_{k-1} = 0.06 < \tau = 0.8$. Fig. 2 illustrates the change process of the state variables of the delay system (3.34) in the time interval [0,1.4].

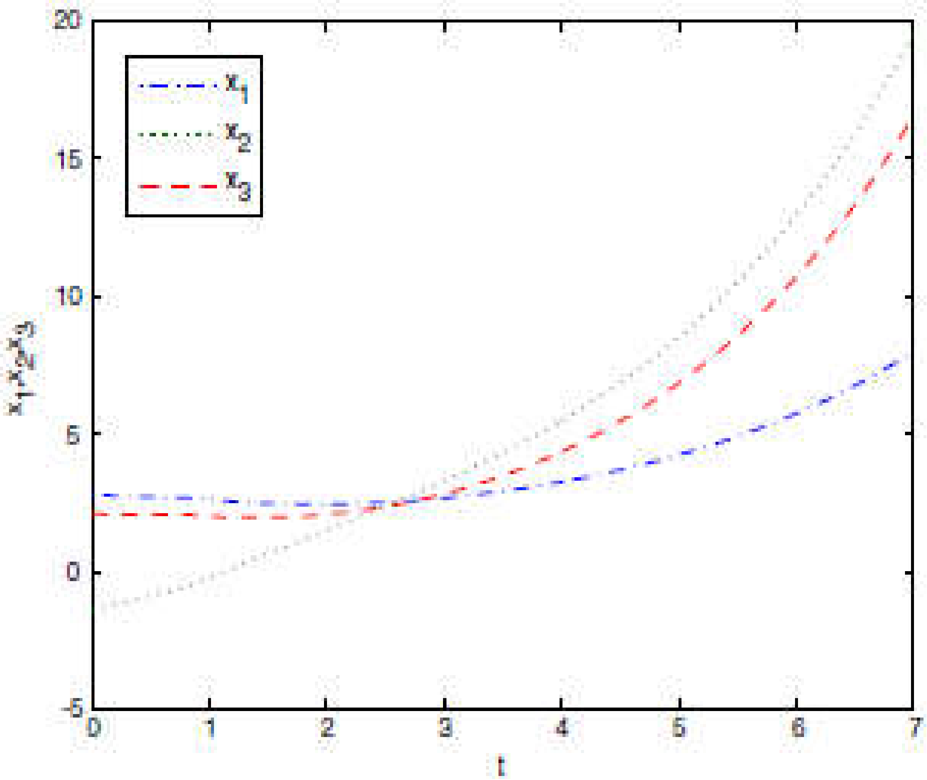

Fig. 1. System without impulses.

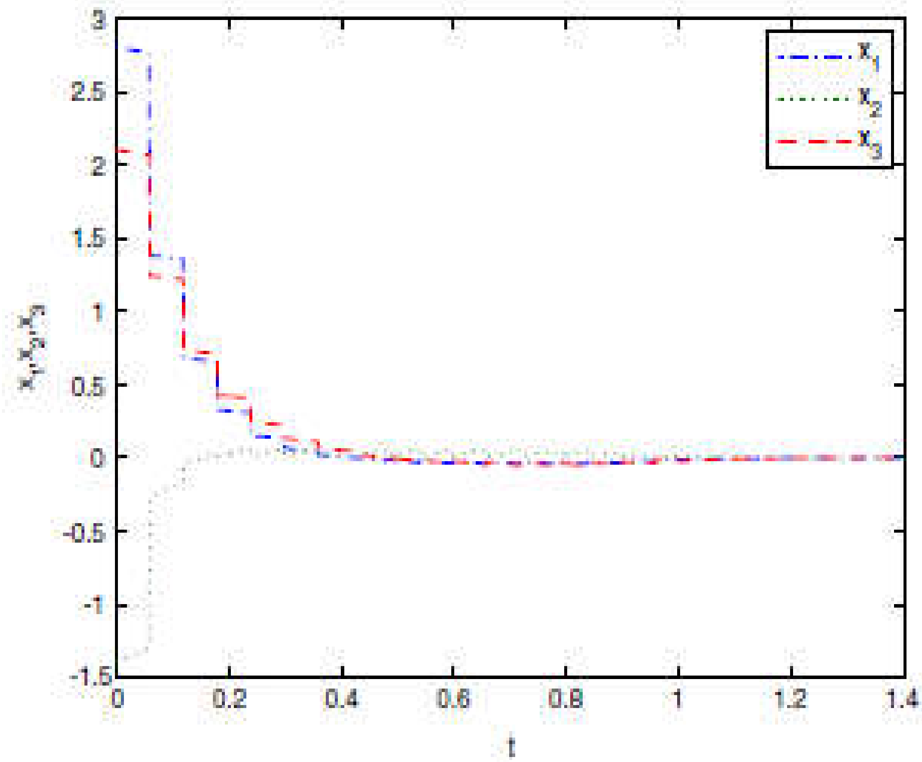

Fig. 2. Impulsively stabilized system.

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