STABILITY, BOUNDEDNESS, AND OSCILLATORY BEHAVIOR OF SECOND ORDER DIFFERENCE EQUATION

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Abstract- The prime objective of the paper is to find the sufficient conditions in terms of coefficients, for stability, boundedness, and oscillatory behavior of the nonlinear difference equation of the form

$$\left(cz_n + dz_{n-1}\right)\Delta_a z_n - bz_n z_{n-1} = 0 \quad ,n \geq 1$$

having the generalized difference operator Δ_a which is defined as $\Delta_a z_n = z_{n+1} - az_n$. Where a,b,c and d are real positive numbers

I. INTRODUCTION

Difference equation is an enchanting subject, in view of the fact that we can derive scores of complex roperties build on simple formulation. Each and every dynamic system $z_{n+1} = f\left(z_n\right)$ reveal a difference equation and reciprocally. There are many interesting dynamic problems in applied science can be formulated by using nonlinear difference equation like population biology, genetics, psychology, electric circuits, vibration of particles …etc. Difference equation has been studied in various branches of mathematics. During the beginning of twentieth centuries', Poincar and Perron obtained first results in qualitative theory of difference equation. Many investigators have concluded about the qualitative behavior of the solution of dynamical system. For example: N Parhi[2] obtained sufficient conditions for nonoscillation of a set of following 3^{rd} third order difference equations.

$$z\left(n+3\right)+\alpha\left(n\right)z\left(n+2\right)+\beta\left(n\right)z\left(n+1\right)+\gamma\left(n\right)z\left(n\right)=0$$

$$\Delta^3 z\left(n-1\right)+a\left(n\right)\Delta^2 z\left(n-1\right)+b\left(n\right)\Delta z\left(n\right)+c\left(n\right)z\left(n\right)=0$$

and

$$\Delta p\left(n-1\right)\ ^2 z\left(n-1\right))+q\left(n\right)\Delta z\left(n\right)+r\left(n\right)z\left(n\right)=0$$

H chen et al.[1] investigates the global stability of the dynamical system $z_{n+1} = \dfrac{Z_n + \alpha Z_{n-1}}{\beta + Z_n}$.

N Parhi and A Panda [3] have studied sufficient conditions for oscillating nature of solutions of following 3^{rd} order nonlinear difference equation. $\Delta_a\left(p_n \Delta_a{}^2 z_n\right)+q_n \Delta_a{}^2 z_n = f(n, z_n\ , \Delta_a z_n\ , \Delta_a{}^2 z_n)$.

In this paper we study stability boundedness, and oscillatory nature of solutions of the following nonlinear 2^{nd} order difference equation. $\left(cz_n + dz_{n-1}\right)\Delta_a z_n - bz_n z_{n-1} = 0 \quad ,n \geq 1$ (1.01)

$n \in N$, where N is the set of positive intigers, a, b, c and $d \in R^{+}$, R is the real numbers set. The generalized difference operator Δ_a, for $a \neq 0$, is defined as $\Delta_a z_n = z_{n+1} - a z_n$. If $a = 1$, we can write $\Delta_1 = \Delta$, the forward difference operator.

A sequence $\{z_n\}$ of real numbers is termed as solution equation (1.01) if it satisfies identically (1.01). Here we consider only nontrivial solutions of (1.01), that is, solution $\{z_n\}$ of (1.01) for which $\sup\{|z_n| : n \geq i\} > 0$ for every $i \in N$.

Let $G : J^{k+1} \to J$ be a function which is continuously differentiable, then for every set of initial conditions $z_{0}, z_{-1}, \ldots, z_{-k} \in J$, the difference equation

$$z_{n+1} = G\left(z_n, z_{n-1}, \ldots z_{n-k}\right),\ n = 0, 1, 2 \ldots \tag{1.02}$$

has a unique solution $\{z_n\}_{n=-k}^{\infty}$ [7]

a) *2. Boundedness*

In this section, sufficient conditions are established for boundedness of equation (1.01).

Definition 2.1. The real sequence $\{z_n\}$ is bounded if $|z_n| \leq k$, $k \in R$ for every $n \in N$.

Theorem 2.2. Let $\{z_n\}$ be solution of equation (1.01), if $a + \dfrac{b}{c+d} \leq 1$, then z_n is bounded.

Proof.

Let us suppose $\{z_n\}$ be solution of equation (1.01)

Expanding equation (1.1) we get

$$\left(c z_n + d z_{n-1}\right) z_{n+1} - \left(c z_n + d z_{n-1}\right) a z_n - b z_n z_{n-1} = 0$$

Or, $$z_{n+1} = a z_n + \frac{b z_n\ z_{n-1}}{c z_n + d\, z_{n-1}} \tag{2.01}$$

Case-I:If $z_n \leq z_{n-1}$,then $z_{n+1} = a z_n + \dfrac{b z_n\ z_{n-1}}{c z_n + d\, z_{n-1}} \leq a z_n + \dfrac{b}{c+d} z_{n-1} \leq \left(a + \dfrac{b}{c+d}\right) z_{n-1}$

Case-II:If $z_{n-1} \leq z_n$,then $z_{n+1} = a z_n + \dfrac{b z_n\ z_{n-1}}{c z_n + d\, z_{n-1}} \leq a z_n + \dfrac{b}{c+d} z_n \leq \left(a + \dfrac{b}{c+d}\right) z_n$

Hence from case-I and II we get

$$z_{n+1} \leq \max\left\{\left(a + \frac{b}{c+d}\right) z_n ,\left(a + \frac{b}{c+d}\right) z_{n-1}\right\} = \left(a + \frac{b}{c+d}\right) \max\{z_n , z_{n-1}\} \tag{2.02}$$

as $a + \dfrac{b}{c+d} \leq 1$, we have $z_{n+1} \leq \max\{z_n , z_{n-1}\} \leq \ldots \leq \max\{z_1 , z_0\}$

Hence z_n is bounded

b) *3. Stability*

In this section, the sufficient condition for stability of equation (1.01) is derived.

Definition 3.1. A point $\bar{z} \in J$ is termed as an equilibrium point of (1.02) if $\bar{z} = (\bar{z}, \bar{z}, \ldots, \bar{z})$,that is $z_n = \bar{z}$, for $n \geq 0$ is a solution of (1.02) or fairly we can say $\bar{z}$ is a fixed point G.

Definition 3.2. If $\lim_{n \to \infty} z_n = \bar{z}$, then the equilibrium point $\bar{z}$ of equation (1.02) is a global attractor.

Definition 3.3. Let f and g be functions $f, g : N \to R^{+}$. $f(n)$ is said to be $O(g(n))$ if $\exists c, n_0 \in I^{+}$ such that for every integer $n \geq n_0$, $f(n) \leq cg(n)$ Where $g(n)$ is an asymptotic upper bound for $f(n)$.

Theorem 3.4.

If $\left(a+\dfrac{b}{c+d}\right)<1$ then 0 is a global attractor of z_n.

Proof:

Let us suppose $\left(a+\dfrac{b}{c+d}\right)<1$. From equation (2.02) we have,

$$0 \leq z_{n+1} \leq \left(a+\frac{b}{c+d}\right)\max\{z_n, z_{n-1}\} \leq \left(a+\frac{b}{c+d}\right)^{2}\max\{z_{n-1}, z_{n-2}\} \leq \ldots$$

$$\leq \left(a+\frac{b}{c+d}\right)^{n}\max\{z_1, z_0\}$$

$$\Rightarrow 0 \leq z_{n+1} \leq \left(a+\frac{b}{c+d}\right)^{n}\max\{z_1, z_0\} \tag{3.01}$$

$$\Rightarrow \lim_{n\to\infty} z_n = 0$$

Hence 0 is the global attractor of z_n

Corollary 3.5. If $\left(a+\dfrac{b}{c+d}\right)<1$, then $z_n = O\left(\left(a+\dfrac{b}{c+d}\right)^{n}\right)$.

Proof:

From equation (3.01) it is clear that $z_n = O\left(\left(a+\dfrac{b}{c+d}\right)^{n}\right)$.

Theorem 3.5. If $\left(a+\dfrac{b}{c+d}\right)=1$, then z_n has global attractor $p>0$.

Proof:

From equation (2.01) we have,

$$z_{n+1} = az_n + (1-a)\frac{(c+d)z_n z_{n-1}}{cz_n + dz_{n-1}} \leq az_n + (1-a)\max\{z_n, z_{n-1}\}$$

$$\leq a\max\{z_n, z_{n-1}\}(1-a)\max\{z_n, z_{n-1}\} \tag{3.02}$$

Let $p_n = \max\{z_n, z_{n-1}\}$, then $p_{n+1} \leq p_n$

Therefore, there exist p_{n_0} such that $p_{n_0} \to p \geq 0$ and $z_{n_0} \leq p_{n_0}$ (3.03)

Hence $\lim supz_{n_0} \leq \lim supp_{n_0} = p$

From (3.02) we have, $p_{n_0+1} \leq az_{n_0} + (1-a)p_{n_0}$

Therefore we get $liminf\ p_{n_0+1} \leq liminf\left\{az_{n_0} + (1-a)p_{n_0}\right\}$

Thus $p \leq a\, liminfz_{n_0} + (1-a)p$

$\Rightarrow \lim supz_{n_0} \leq p \leq liminfz_{n_0}$

Therefore $z_n \to p$

c) 4. Oscillatory

In this section, sufficient conditions are found for oscillatory nature of equation (1.01)

Definition 4.1. Let $\{z_n\}$ be the solution of (1.01) said to be nonoscillatory, if all the terms of $\{z_n\}$ are either eventually positive or eventually negative for $n > N$, $N \in I^+$. Otherwise $\{z_n\}$ is oscillatory.

Theorem4.2. If $\left(a + \frac{b}{c+d}\right) \le 0$, then all solution of equation (1.01) are oscillatory.

Proof:

If possible let $\{z_n\}$ be a nonoscillatory solution of (1.01).

Hence for every $n \ge N$, let $\{z_n\}$ be eventually positive or eventually negative. Assume $\{z_n\}$ is eventually positive for $n \ge N$.

That is $z_n > 0, z_{n+1} > 0, \ldots$

From equation (2.01) we have, $z_n z_{n+1} = a z_n^{\ 2} + \frac{b z_n^{\ 2} z_{n-1}}{c z_n + d z_{n-1}}$ (4.01)

Case-I: Let $z_n \le z_{n-1}$, then $z_n z_{n+1} \le \left(a + \frac{b}{c+d}\right) z_{n-1}^{\ 2}$

Case-II: Let $z_{n-1} \le z_n$, then $z_n z_{n+1} \le \left(a + \frac{b}{c+d}\right) z_n^{\ 2}$

For both cases $z_n z_{n+1} \le 0$. This is a contradiction to our assumption. Similarly we can establish it, if $\{z_n\}$ is eventually negative.

$\therefore$ The theorem.

Corollary 4.3. If $\left(a + \frac{b}{c+d}\right) > 0$, then all solutions of equation (1.01) are nonoscillatory.

Proof:

From both cases of the above theorem it is clear that $z_n z_{n+1} > 0$, if $\left(a + \frac{b}{c+d}\right) > 0$.

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