A Common Fixed Point Theorem for Generalized Weakly Contractive mappings in metric spaces Satisfying Implicit Relations

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Abstract:The main aim of this paper, motivated and inspired by Samet et al. [25], we introduce the notion of generalized weakly contractive mappings in metric spaces and prove the existence and uniqueness of fixed point for such mappings, and we obtain a coupled fixed point theorem in metric spaces. These theorems generalize many previously obtained fixed point results. An example is given to illustrate the main result. Finally, we give applications of our results to fixed point results in partial metric spaces.

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1.**Introduction and preliminaries** :

Throughout this paper □ and □ represent the set of real numbers and the set of natural numbers respectively. It is well known that Banach's contraction principle is one of the pivotal results of metric fixed point theory. Banach's contraction principle $[1]$ states that if (X,d) is a complete metric space and T: $X \to X$ is a self mapping such that $d(Tx,Ty) \le \alpha d(x,y)$,for all x,y $\in X$, $where 0 \le \alpha \le 1$,then T has a unique fixed point. This theorem ensures the existence uniqueness of fixed points of certain self maps of metric spaces and it gives a useful contractive method to find those fixed points. In 1977, Alber et al. [3] generalized Benach's contraction principle by introducing the concept of weak contraction mapping in Hilbert spaces. Weak contraction principle states that every weak contraction mapping on a complete Hilbert space has a unique fixed point. Rhoades [4] extended weak contraction principle. Khan et al. [17] obtained fixed point theorems in metric spaces by introducing the concept of altering distance functions.

In particular, Choudhury et al. [10] obtained a generalization of weak contraction principle in metric spaces by using altering distance functions as follows.

Theorem 1.1 ([10]): Suppose that a mapping $g : X \to X$ where X is a metric space with metric d, satisfies the following condition.

$$\psi\left(d\left(gx, gy\right)\right) \le \psi\left(\max\left\{d(x,y),\, d(x,gx), d(y,gy),\, \frac{1}{2}\left\{d(x,gy)+d(y,gx)\right\}\right\}\right) - \phi\left(\max\left\{d(x,y), d(y,gy)\right\}\right) \quad (1.1)$$

for all $x, y \in X$, where $\phi:[0,\infty) \to [0,\infty)$ is a continuous function, and $\psi:[0,\infty) \to [0,\infty)$ is an altering distance functions, that is, ψ is a nondecreasing and continuous function, and $\psi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Matthews [18] introduced the notion of partial metric spaces, and extended Banach's contraction principle to partial metric spaces, and then a lot of authors gave fixed point results in partial metric spaces (see [7, 19-23]). Also, Aydi et al. [24] extended Ekeland's variational principle to partial metric spaces, and Aydi et al, [21] extended Caristi's fixed point theorem to partial metric spaces.

In particular, Abdeljawad [3] extended the result of Choudhury et al. [10] to partial metric spaces.

Samet et al.[25] gave a generalization of Banach's contraction principle and an application to fixed point results in partial metric spaces.

In this paper, motivated and inspired by Samet et al. [25], we introduce the notion of generalized weakly contractive mappings in metric spaces and prove a fixed point theorem for generalized weakly contractive mappings defined on complete metric spaces, which is generalization of the results of [2,10 –12, 26]. Also, we obtain a coupled fixed point theorem in metric spaces by applying our main result, and we give applications to fixed point and coupled fixed point theorems in partial metric spaces.

A function $f: X \to [0,\infty)$, where X is a metric space, is called lower semi continuous if, for all $x \in X$ and $\{x_n\} \subset X$ with $\lim_{n\to\infty} x_n = x$, we have

$$f(x) \le \lim_{n\to\infty} \inf f(x_n)$$

Let

$$\Psi = \{\psi:[0,\infty) \mid \psi \text{ is continuous and } \psi(t)=0 \Leftrightarrow t=0\}$$

Also, we denote

$$\varphi = \{\phi:[0,\infty) \mid [0,\infty) \mid \phi \text{ is lower semicontinuous and } \phi(t)=0 \Leftrightarrow t=0\}$$

Lemma 1.1 ([27]) If a sequence $\{x_n\}$ in X is not Cauchy, then there exist $\in > 0$ and two subsequences $\{x_{m(k)}\}$ of $\{x_{n(k)}\}$ such that *m(k)* is the smallest index for which *m(k) > n(k) > k,*

$$d\left(x_{m(k)}, x_{n(k)}\right) \ge \in, \tag{1.2}$$

and

$$d\left(x_{m(k)-1}, x_{n(k)}\right) < \in \tag{1.3}$$

Moreover, suppose that .
Then we have,

(1) $\lim_{n\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \in;$

(2) $\lim_{n\to\infty} d\left(x_{m(k)-1}, x_{n(k)-1}\right) = \in;$

(3) $\lim_{n\to\infty} d\left(x_{m(k)}, x_{n(k)-1}\right) = \in;$

(4) $\lim_{n\to\infty} d\left(x_{m(k)-1}, x_{n(k)}\right) = \in;$

2. Fixed point results

Let X be a metric space with metric d, let $S: X \to X$ and let $\varphi: X \to [0, \infty)$ be a lower semi continuous function.

Then S is called a generalized weakly contractive mapping if it satisfies the following condition. :

$$\psi\left(d(Sx,Sy)+\varphi(Sx)+\varphi(Sy)\right)$$

$$\leq \psi\big(m(x,y,d,S,\psi)\big) - \phi\big(l(x,y,d,S,\varphi)\big) \ \forall\ x,y \in X, \tag{2.1}$$

Where $\psi \in \Psi$, $\phi \in \Phi$ and

$$m(x,y,d,S,\varphi) = \max\{d(x,y)+\varphi(x)+\varphi(y), d(x,Sx)+\varphi(x)+\varphi(Sx),$$
$$d(y,Sy)+\varphi(y)+\varphi(Sy), \frac{1}{2}\{d(x,Sy)+\varphi(x)+\varphi(Sy) \tag{2.2}$$
$$+ d(y,Sx)+\varphi(y)+\varphi(Sx)\}\},$$

and

$$l(x,y,d,S,\varphi) = \max\{d(x,y)+\varphi(x)+\varphi(y), d(y,Sy)+\varphi(y)+\varphi(Sy)\} \tag{2.3}$$

Let X be a metric space with metric d, let $S: X \to X$, and let $\varphi: X \to [0,\infty)$ be a lower semi continuous function.

Theorem 2.1 Let X be complete. If S is a generalized contractive mapping, then there exists a unique $z \in X$ such that z = Sz and $\varphi(z) = 0$.

Proof: Let $x_0 \in X$ be a fixed point, and define a sequence $\{x_n\}$ by $x_{n+1} = Sx_n$ for all n = 0, 1,2,……

If $x_n = x_{n+1}$ for some n, then $x_n = x_{n+1} = Sx_n$, so x_n is a fixed point of S, and the proof is finished.

From now on, assume that $x_n \neq x_{n+1}$ for all n =0,1,2,…..

From (2.2) with $x = x_{n-1}$ and $y = x_n$ we have

$$m(x_{n-1},x_n,S,\varphi)$$
$$= \max\{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n),$$
$$d(x_{n-1},Sx_{n-1})+\varphi(x_{n-1})+\varphi(Sx_{n-1}), d(x_n,Sx_n)+\varphi(x_n)+\varphi(Sx_n),$$
$$\frac{1}{2}\{d(x_{n-1},Sx_n)+\varphi(x_{n-1})+\varphi(Sx_n)+d(x_n,Sx_{n-1})+\varphi(x_n)+\varphi(Sx_{n-1})\}\}$$

Since

$$\frac{1}{2}\{d(x_{n-1},Sx_n)+\varphi(x_{n-1})+\varphi(Sx_n)+d(x_n,Sx_{n-1})+\varphi(x_n)+\varphi(Sx_{n-1})\}$$
$$= \frac{1}{2}\{d(x_{n-1},x_{n+1})+\varphi(x_{n-1})+\varphi(x_{n+1})+d(x_n,x_n)+\varphi(x_n)+\varphi(x_n)\}$$
$$\leq \frac{1}{2}\{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)+d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})\}$$
$$\leq \max\{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n), d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})\},$$

We obtain

$$m(x_{n-1},x_n,d,S,\varphi)$$
$$= \max\{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n), d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})\} \tag{2.4}$$

Also, we have

$$l(x_{n-1},x_n,d,S,\varphi)$$
$$= \max\{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n), d(x_n,Sx_n)+\varphi(x_n)+\varphi(Sx_n)\}$$
$$= \max\{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n), d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})\}$$

It follows from (2.1) that

$$\psi\left(d\left(x_n, x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\right)$$
$$=\psi\left(d\left(Sx_{n-1}, Sx_n\right)+\varphi\left(Sx_{n-1}\right)+\varphi\left(Sx_n\right)\right) \quad (2.5)$$
$$\leq \psi\left(m\left(x_{n-1}, x_n, d, S, \varphi\right)\right)-\phi\left(l\left(x_{n-1}, x_n, d, S, \varphi\right)\right).$$

If

$$d\left(x_{n-1}, x\ \right)+\varphi\left(x_{n-1}\right)+\varphi\left(x_n\right)<d\left(x_n, x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)$$

For some positive integer n, then from (2.5) we obtain

$$\psi\left(d\left(x_n, x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\right)$$
$$\leq \psi\left(d\left(x_n, x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\right)-\phi\left(d\left(x_n, x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\right)$$

Which implies

$$\phi\left(d\left(x_{n+1}, x_n\right)+\varphi\left(x_{n+1}\right)+\varphi\left(x_n\right)\right)=0$$

and so

$$d\left(x_{n+1}, x_n\right)+\varphi\left(x_{n+1}\right)+\varphi\left(x_n\right)=0$$

Hence

$$x_{n+1}=x_n \text{ and } \varphi\left(x_{n+1}\right)=\varphi\left(x_n\right)=0$$

Which is a contradiction.

Thus we have

$$d\left(x_n, x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right) \leq d\left(x_{n-1}, x_n\right)+\varphi\left(x_{n-1}\right)+\varphi\left(x_n\right) \quad (2.6)$$

for all n = 1, 2, 3, ……, and so

$$m\left(x_{n-1}, x_n, d, S, \varphi\right)=d\left(x_{n-1}, x_n\right)+\varphi\left(x_{n-1}\right)+\varphi\left(x_n\right)$$

and

$$l\left(x_{n-1}, x_n, d, S, \varphi\right)=d\left(x_{n-1}, x_n\right)+\varphi\left(x_{n-1}\right)+\varphi\left(x_n\right)$$

For all n = 1,2,3, ……

It follow from (2.5) that

$$\psi\left(d\left(x_n, x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\right)$$
$$\leq \psi\left(d\left(x_{n-1}, x_n\right)+\varphi\left(x_{n-1}\right)+\varphi\left(x_n\right)\right) \quad (2.7)$$
$$-\phi\left(d\left(x_{n-1}, x_n\right)+\varphi\left(x_{n-1}\right)+\varphi\left(x_n\right)\right)$$

It follows from (2.6) that the sequence $\left\{d\left(x_n, x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right)\right\}$ is non increasing.

Thus we have

$$d\left(x_n, x_{n+1}\right)+\varphi\left(x_n\right)+\varphi\left(x_{n+1}\right) \rightarrow r \quad as\ n \rightarrow \infty$$

For some $r \geq 0$.

Assume that r > 0.

Letting $n \rightarrow \infty$ in (2.7), by the continuity of ψ and the lower semi continuity of ϕ it follows that

$$\psi(r) \leq \psi(r)-\lim_{n \rightarrow \infty} \inf \phi\left(dx_{n-1}, x_n\right)+\varphi\left(x_{n-1}\right)+\varphi\left(x_n\right))$$
$$\leq \psi(r)-\phi(r)$$

Since $r>0, \phi(r)>0$. Hence

$$\psi(r) \leq \psi(r)-\phi(r)<\psi(r)$$

a contradiction .

Hence,

$$\lim_{n\to\infty}\{d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})\}=0$$

Which implies

$$\lim_{n\to\infty} d(x_n,x_{n+1})=0 \quad (2.8)$$

and

$$\lim_{n\to\infty} \varphi(x_n)=0 \quad (2.9)$$

Now, we prove that the sequence is Cauchy.

If is not Cauchy, then by Lemma 1.1 there exist ε and subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of such that (1.2) and (1.3) hold.

From (2.2) we have

$$m(x_{n(k)},x_{m(k)},d,S,\varphi)$$
$$=\max\{d(x_{n(k)},x_{m(k)})+\varphi(x_{n(k)})+\varphi(x_{m(k)}),$$
$$\{d(x_{n(k)},Sx_{n(k)})+\varphi(x_{n(k)})+\varphi(Sx_{n(k)}),d(x_{m(k)},Sx_{m(k)})+\varphi(x_{m(k)})+\varphi(Sx_{n(k)}),$$
$$\frac{1}{2}\{d(x_{n(k)},Sx_{m(k)})+\varphi(x_{n(k)})+\varphi(Sx_{m(k)})+d(x_{m(k)},Sx_{n(k)})+\varphi(x_{m(k)})+\varphi(Sx_{n(k)})\}\}$$
$$=\max\{d(x_{n(k)},x_{m(k)})+\varphi(x_{n(k)})+\varphi(x_{m(k)}),d(x_{n(k)},x_{n(k)+1})+\varphi(x_{n(k)})+\varphi(x_{n(k)+1}),$$
$$d(x_{m(k)},x_{m(k)+1})+\varphi(x_{m(k)})+\varphi(x_{m(k)+1}),\frac{1}{2}\{d(x_{n(k)},x_{m(k)+1})+\varphi(x_{n(k)})+\varphi(x_{m(k)+1})$$
$$+d(x_{m(k)},x_{n(k)+1})+\varphi(x_{m(k)})+\varphi(x_{n(k)+1})\}\} \quad (2.10)$$

Letting $k\to\infty$ in (2.10) and applying Lemma 1.1, (2.8), and (2.9), it follows that

$$\lim_{k\to\infty} m(x_{n(k)},x_{m(k)},d,S,\varphi)=\in \quad (2.11)$$

Also, it follows from (2.3) that

$$l(x_{n(k)},x_{m(k)},d,S,\varphi)$$
$$=\max\{d(x_{n(k)},x_{m(k)})+\varphi(x_{n(k)})+\varphi(x_{m(k)}),d(x_{m(k)},Sx_{m(k)+1})+\varphi(x_{m(k)})+\varphi(Sx_{m(k)}),$$
$$=\max\{d(x_{n(k)},x_{m(k)})+\varphi(x_{n(k)})+\varphi(x_{m(k)}),d(x_{m(k)},x_{m(k)+1})+\varphi(x_{m(k)})+\varphi(x_{m(k)+1}).$$

Hence,

$$\lim_{k\to\infty} l(x_{n(k)},x_{m(k)},d,S,\varphi)=\in \quad (2.12)$$

From (2.1) we have

$$\psi(d(x_{n(k)},x_{m(k)+1})+\varphi(x_{n(k)+1})+\varphi(x_{m(k)+1}))$$
$$\le\psi(m(x_{n(k)},x_{m(k)},d,S,\varphi))-\phi(l(x_{n(k)},x_{m(k)},d,S,\varphi)).$$

Letting $k\to\infty$ in this inequality, by Lemma 1.1, the continuity of ψ, the lower semi continuity of ϕ, and by (2.9), (2.11), and (2.12) we have

$$\psi(\in)\le\psi(\in)-\phi(\in),$$

Which is a contradiction because $\phi(\in)>0$.

Hence the sequence $\{x_n\}$ is Cauchy, and Hence

$$\lim_{n\to\infty} x_n=z\in X \text{ exists}$$

Because X is complete. Since φ is lower semi continuous.

$$\varphi(z) \le \lim_{n\to\infty} \inf \varphi(x_n) \le \lim_{n\to\infty} \varphi(x_n) = 0,$$

Which implies

$$\varphi(z) = 0 \tag{2.13}$$

It follows from (2.2) that

$$\begin{aligned} m(x_n, z, d, S, \varphi) &= \max\{d(x_n, z) + \varphi(x_n) + \varphi(z), \\ &\quad d(x_n, Sx_n) + \varphi(x_n) + \varphi(Sx_n), d(z, Sz) + \varphi(z) + \varphi(Sz), \\ &\quad \frac{1}{2}\{d(x_n, Sz) + \varphi(x_n) + \varphi(Sz) + d(z, Sx_n) + \varphi(z) + \varphi(Sx_n)\}\} \\ &= \max\{d(x_n, z) + \varphi(x_n) + \varphi(z), \\ &\quad d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}), d(z, Sz) + \varphi(z) + \varphi(Sz), \\ &\quad \frac{1}{2}\{d(x_n, Sz) + \varphi(x_n) + \varphi(Sz) + d(z, x_{n+1}) + \varphi(z) + \varphi(x_{n+1})\}\}. \end{aligned}$$

So, we have

$$\lim_{n\to\infty} m(x_n, z, d, S, \varphi) = d(z, Sz) + \varphi(z) + \varphi(Sz) = d(z, Sz) + \varphi(Sz)$$

(2.14)

Also, we have

$$\begin{aligned} \lim_{n\to\infty} l(x_n, z, d, S, \varphi) &= \lim_{n\to\infty} \max\{d(x_n, z) + \varphi(x_n) + \varphi(z), d(z, Sz) + \varphi(z) + \varphi(Sz)\} \\ &= d(z, Sz) + \varphi(z) + \varphi(Sz) = d(z, Sz) + \varphi(Sz) \end{aligned} \tag{2.15}$$

It follows from (2.1) that

$$\begin{aligned} \psi(d(x_{n+1}, Sz) + \varphi(x_{n+1}) + \varphi(Sz)) &= \psi(d(Sx_n, Sz) + \varphi(Sx_n) + \varphi(Sz)) \\ &\le \psi(m(xn, z, d, S, \varphi) - \phi(l(x_n, z, d, S, \varphi))) \end{aligned} \tag{2.16}$$

By taking the limit as $n \to \infty$ in (2.16) and by applying the continuity of ψ, the lower semi continuity of ϕ, (2,14) and (2.15) we have

$$\psi(d(z, Sz) + \varphi(Sz)) \le \psi(d(z, Sz) + \varphi(Sz)) - \phi(d(z, Sz) + \varphi(Sz))$$

Hence $d(z, Sz) + \varphi(Sz) = 0$, and hence $z = Sz$ and $\varphi(Sz) = 0$.

Suppose that u is another fixed point of S.

Then $u = Su$ and $\varphi(u) = 0$

By applying (2.1) with $x = z$ and $y = u$ we have

$$\begin{aligned} \psi(d(z, u)) &= \psi(d(Sz, Su)) \\ &= \psi(d(Sz, Su) + \varphi)(Sz) + \phi(Su)) \\ &= \psi(m(z, u, d, S, \varphi)) - \phi(l(z, u, d, S, \psi)) \\ &= \psi(d(z, u)) - \phi(d(z, u)) \end{aligned}$$

Which implies z =u.

The following example illustrates Theorems 2.1 and shows that it is a real generalized of Theorem 3.1 in [10].

Example 2.1 Let $X = [0, \infty)$ and $d(x, y) = |x - y|$ for $x, y \in X$, let $\psi(t) = \frac{3}{2}t$ for $t \ge 0$, and let

$$\varphi(t)=\begin{cases}\frac{1}{2}t & (0\le t\le 1),\\ \frac{1}{2}t+\frac{1}{2} & (1<t\le 2),\\ t & (t>2).\end{cases}$$

Then $\psi\in\Psi$, φ is lower semi continuous, and $\frac{1}{2}t\le\varphi(t)\le t,\ t\ge 0.$.

Define the map $S: X\to X$ by

$$Sx=\frac{x^2}{2(1+x)}.$$

Assume that a function $\phi:[0,\infty)\to[0,\infty)$ is defined by

$$\phi(t)=\frac{3t}{4+2t}.$$

Then $\phi\in\Phi$.

We now show that (2.1) holds.

Without loss of generality, suppose that $x\ge y$.

Then we have

$$\frac{1}{2}\{d(x,Sy)+\varphi(x)+\varphi(Sy)+d(y,Sx)+\varphi(y)+\varphi(Sx)\}$$

$$\ge\frac{1}{2}\left\{d(x,Sy)+\frac{1}{2}x+\frac{1}{2}Sy+d(y,Sx)+\frac{1}{2}y+\frac{1}{2}Sx\right\}$$

$$\ge\left\{\frac{1}{2}\{d(x,Sy)+x+Sy+d(y,Sx)+y+Sx\}\right\}$$

$$=\begin{cases}\frac{1}{2}\left(x+\frac{x^2}{1+x}\right) & \left(y\le\frac{x^2}{2(1+x)}\right)\\ \frac{1}{2}(x+y) & otherwise\end{cases}$$

$$>\frac{1}{2}x.$$

Thus we have

$$m(x,y,d,S,\varphi)$$

$$=\max\{d(x,y)+\varphi(x)+\varphi(y),d(x,Sx)+\varphi(x)+\varphi(Sx),$$

$$d(y,Sy)+\varphi(y)+\varphi(Sy),\frac{1}{2}\{d(x,Sy)+\varphi(x)+\varphi(Sy)+d(y,Sx)+\varphi(y)+\varphi(Sx)\}\}$$

$$\ge\frac{1}{2}\max\{d(x,y)+x+y,d(x,Sx)+x+Sx,$$

$$d(y,Sy)+y+Sy,\frac{1}{2}\{d(x,Sy)+x+Sy+d(y,Sx)+y+Sx\}\}$$

$$=\frac{1}{2}\max\left\{2x,\ 2x,\ 2y,\ \frac{1}{2}x\right\}$$

$$=x.$$

And

$$\begin{aligned} l(x,y,d,S,\varphi) &= \max\{d(x,y)+\varphi(x)+\varphi(y), d(y,Sy)+\varphi(y)+\varphi(Sy)\} \\ &\le \max\{d(x,y)+x+y, d(y,Sy)+y+Sy\} \\ &= \max\{2x, 2y\} \\ &= 2x. \end{aligned}$$

Also, we have

$$\begin{aligned} \psi(d(Sx,Sy)+\varphi(Sx)+\varphi(Sy)) \le \psi(d(Sx,Sy)+Sx+Sy) \\ &= \frac{3}{2}\left(\left|\frac{x^2}{2(1+x)}-\frac{y^2}{2(1+y)}\right|+\frac{x^2}{2(1+x)}+\frac{y^2}{2(1+y)}\right) \\ &= \frac{3}{2}\cdot\frac{2x^2}{2(1+x)} \\ &= \frac{3}{2}\cdot\frac{x^2}{1+x} \end{aligned}$$

Hence,

$$\begin{aligned} \psi(m(x,y,d,S,\varphi)) - \varphi(l(x,y,d,S,\varphi)) &\ge \frac{3}{2}x - \frac{3x/2}{1+x} \\ &= \frac{3}{2}\cdot\frac{x^2}{1+x} \\ &\ge \psi(d(Sx,Sy)+\varphi(Sx)+\varphi(Sy)) \end{aligned}$$

Where the equality is satisfied when x = 0.

Thus (2.1) is satisfied.

By Theorem 2.1, S has a unique fixed point z = 0, and $\varphi(z)=0$.

However, (1.1) is not satisfied. In fact, let x = *3*, *y* = *1* and $\varphi(t)=0, t\ge 0$.

Then

$$\psi\left(m((x,\ y,\ d,\ S,\ \varphi))\right)=\frac{45}{2},$$

$$\phi\left(l((x,y,d,\ S,\ \varphi))\right)=\frac{3}{4},$$

$$\psi\left(d(Sx,\ Sy)\right)=\frac{51}{2},$$

and so

$$\psi(d(Sx,Sy))=\frac{204}{8}>\frac{147}{8}=\psi(m((x,y,d,S,\varphi)))-\varphi(l((x,y,d,S,\varphi)))$$

The proofs of the following Corollary 2.2 and Corollary 2.3 are similar to that of Theorem 2.1. So, here the proofs are omitted.

Corollary 2.2 Let X be complete. Suppose that S satisfies the following condition :

$$\begin{aligned} &\psi(d(Sx,Sy)+\varphi(Sx)+\varphi(Sy)) \\ &\le (m(x,y,d,S,\varphi)) - \phi(l(x,y,d,S,\varphi)) \\ &\forall\ x,y\in X,\ \textit{where}\ \psi\in\Psi\ \textit{and}\ \phi\in\Phi. \end{aligned}$$

Then there exists a unique $z \in X$ such that z = Sz and $\varphi(z) = 0$.

Corollary 2.3 Let (X,d) be complete. Suppose that S satisfies the following condition:

$$\psi\left(d\left(Sx,Sy\right) + \varphi\left(Sx\right) + \varphi\left(Sy\right)\right)$$
$$\leq\left(d\left(x,y\right)+\varphi\left(x\right) + \varphi\left(y\right)\right) - \phi\left(d\left(x,y\right) + \varphi\left(x\right) + \varphi\left(y\right)\right)$$
$$\forall\ x,y\in X,\ where\ \psi\in\Psi\ and\ \phi\in\Phi.$$

Then there exists a unique $z \in X$ such that z = Sz and $\varphi(z) = 0$.

Corollary 2.4 Let X be complete. Suppose that S satisfies the following condition:

$$\psi\left(d\left(S^k x,S^k y\right) + \varphi\left(S^k x\right) + \varphi\left(S^k y\right)\right)$$
$$\leq \psi\left(m\left(x,y,d,S^k,\varphi\right)\right) - \phi\left(l\left(x,y,d,S^k,\varphi\right)\right)$$
$$\forall\ x,y\in X,\ where\ \psi\in\Psi\ ,\phi\in\Phi\ and\ k\ is\ a\ positive\ integer.$$

Then there exists a unique $z \in X$ such that z = Sz and $\varphi(z) = 0$.

Proof: Let T = S^k. Then by Theorem 2.1 T has a unique fixed point, say z.
Then $S^k z$ = Tz = z and

$$\psi\left(z\right) = \varphi\left(Tz\right) = \varphi\left(S^k z\right) = 0$$

Since $S^{k+1} z = Sz$.

$$TSz = S^k\left(Sz\right) = S^{k+1} z = Sz,$$

and so Sz is a fixed point of T. By the uniqueness of a fixed point of T, Sz = z.

Remark 2.1 If we have $\varphi = 0$, then ψ is nondecreasing and continuous, and ϕ is continuous in Theorem 2.1 (resp. Corollary 2.3, Corollary 2.4), then we obtain Theorem 3.1 of [10] (resp. Theorem 2.1 of {12}, Corollary 3.1 of [10]}.

Remark 2.2 If $\varphi = 0$ and if ψ and ϕ are nondecreasing and continuous in Corollary 2.3, then we obtain Theorem 2.1 of [12].

Remark 2.3 If $\varphi = 0$ and ψ is nondecreasing and continuous in Corollary 2.2, then we obtain Theorem 2.2 of [11].

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