THERMOELASTIC SOLUTION OF RECTANGULAR PLATE WITH MOVING HEAT SOURCE: INVERSE PROBLEM

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ABSTRACT- In this paper thermoelastic solution of rectangular plate subjected to the activity of a moving heat source is presented. Here the temperature distribution and thermal stresses with the help of integral transform technique have been derived. The results are obtained in term of Bessel's function in the form of infinite series.

KEY WORDS*:* Moving heat source, Marchi-Fasulo transform, Fourier Cosine Transform. rectangular plate, inverse problem.

I. INTRODUCTION

Araya et al. [1] have derived analytical solution for a transient three dimensional temperature distribution due to a moving laser beam. **Cheng et al. [**2] have studied an analytical model for the temperature field in the laser forming of a sheet metal. **Khobragade et al.** [3] have discussed inverse unsteady-state thermoelastic problem of a thin rectangular plate. **Hiranwar et al.** [4] have investigated thermal deflection of a thick clamped rectangular plate. **Kidawa-Kukla** [5] has studied temperature distribution in a rectangular plate heated by a moving heat source. **Chapke et al.** [6] have discussed thermal stresses of a circular plate with internal heat source. **Marchi and Fasulo** [7] have studied heat conduction in sector of hollow cylinder with radiation.

Patil et al. [11] have studied direct thermoelastic problem of heat conduction with internal heat generation and partially distributed heat supply in rectangular plate. **Roy et al.** [12] have discussed transient thermoelastic problem of an infinite rectangular slab. **Bagade et al.** [13] have derived thermal stresses of a semi infinite rectangular beam. **Solanke et al.** [14] have discussed quasi-static transient stresses in a Neumann's thin rectangular plate with internal moving heat source and **Durge et al.**[15] have studied quasi-static thermal stresses in thin rectangular plate with internal moving line heat source. **Sutar et al.** [16] have discussed inverse thermoelastic problem of heat conduction with internal heat generation for the rectangular plate. **Thakare et al.** [18] have derived thermal stresses of a thin rectangular plate with internal moving heat source.

In present paper, authors have considered thermoelastic problem with first, second and third kind boundary condition on a rectangular plate occupying the region $D: -a \le x \le a,\ 0 \le y \le b, 0 \le z < h\}$. The solution of the problem is obtained by using finite Marchi-Fasulo transform and Fourier cosine transform techniques. The results are obtained in terms of Bessel's function in the form of infinite series.

2. STATEMENT OF THE PROBLEM

Consider semi-infinite rectangular beam occupying the region $D: -a \le x \le a,\ 0 \le y \le b, 0 \le z < h\}$. The beam is subjected to the motion of moving point heat source at the point $(0, y', z')$ which move its place along x, y, z axes with constant velocity vector $\bar{v} = v_1 i + v_2 j + v_3 k$ where v_1, v_2, v_3 are component of velocity vector along x, y, z axes respectively. The temperature distribution of the rectangular beam is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{g(x, y, z, t)}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.1}$$

where k is the thermal conductivity and α is thermal diffusivity of the material of the plate.

Consider an instantaneous moving point heat source at point $(0, y', z')$ and releasing its heat spontaneously at time t'. Such volumetric moving heat source in rectangular coordinates is given by

$$g(x,y,z,t)=g_0\delta(x)\delta(y-y')\delta(z-z')\delta(t-t')$$

Hence equation (2.1) becomes

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{1}{k}g_0\delta(x)\delta(y-y')\delta(z-z')\delta(t-t')=\frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.2}$$

where $y'=v_2 t$ and $z'=v_3 t$,

With initial condition

$$T(x,y,z,0)=0 \tag{2.3}$$

And the boundary conditions are given by

$$\left[T(x,y,z,t)+k_1\frac{\partial T(x,y,z,t)}{\partial x}\right]_{x=a}=0 \tag{2.4}$$

$$\left[T(x,y,z,t)+k_2\frac{\partial T(x,y,z,t)}{\partial x}\right]_{x=-a}=0 \tag{2.5}$$

$$\left[\frac{\partial T}{\partial y}\right]_{y=0}=0 \tag{2.6}$$

$$\left[\frac{\partial T}{\partial y}\right]_{y=\xi}=0 \quad \text{(known)} \tag{2.7}$$

$$[T]_{y=b}=F(x,z,t) \text{ (unknown)} \tag{2.8}$$

$$[T(x,y,z,t)]_{z=0}=f_1(x,y,t) \tag{2.9}$$

$$[T(x,y,z,t)]_{z=h}=f_2(x,y,t) \tag{2.10}$$

Introduce a thermal stress function χ related to component of stress in the rectangular coordinates system as [5] is

$$\chi=\chi_c+\chi_p \tag{2.11}$$

where χ_c is the complementary solution and χ_p is particular solution.

χ_c and χ_p are governed by equations:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2\chi_c=0 \tag{2.12}$$

and

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2\chi_p=-\alpha E\Gamma . \tag{2.13}$$

where $\Gamma=T-T_0$, T_0 is initial temperature. Since the plate is thin, z is negligible.

The stress functions are given by

$$\sigma_{xx}=\frac{\partial^2\chi}{\partial y^2} \tag{2.14}$$

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$$\sigma_{yy} = \frac{\partial^2 \chi}{\partial x^2} \tag{2.15}$$

$$\sigma_{xy} = -\frac{\partial^2 \chi}{\partial x \partial y} \tag{2.16}$$

And $\sigma_{yy} = 0$, $\sigma_{xy} = 0$ at $y = b$.

Equations (2.1) to (2.16) constitute the mathematical formulation of the problem under consideration.

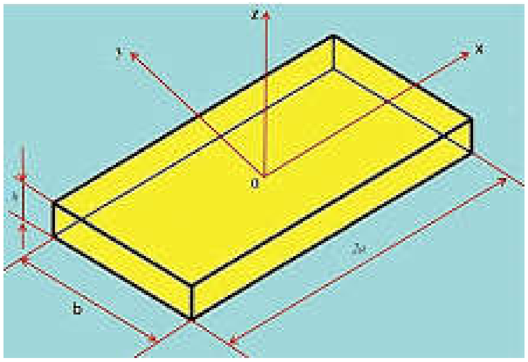

Figure 1: Geometry of the problem

3. SOLUTION OF THE PROBLEM

Applying finite Marchi-Fasulo transform, finite Fourier sine transform and finite Fourier cosine transform, we get

$$\frac{d\overline{\overline{T}}^*}{dt} + \alpha p^2 \overline{\overline{T}}^* = \Psi \tag{3.1}$$

Where $p^2 = \lambda_l^2 + \frac{n^2\pi^2}{h^2} + \frac{m^2\pi^2}{\xi^2}$

$$\Psi = \alpha\left[\frac{g_0}{k} P_l(0) \sin\left(\frac{n\pi z'}{h}\right)\cos\left(\frac{m\pi y'}{\xi}\right)\delta(t-t')\right]$$

Solving above equation and using initial condition we get

$$\overline{\overline{T}}^* = e^{-\alpha p^2 t}\left[\overline{\overline{T}}^*_0 + \int_0^t \Psi e^{\alpha p^2 \tau} d\tau\right] \tag{3.2}$$

Taking inverse Fourier cosine transform, finite Fourier sine and Marchi-Fasulo transform, we get

$$T = \left(\frac{4}{\xi h}\right)\sum_{l,m,n,=1}^{\infty} \frac{P_l(x)}{\lambda_l} \sin\left(\frac{n\pi z}{h}\right)\cos\left(\frac{m\pi y}{\xi}\right) \times e^{-\alpha p^2 t}\left[\int_0^t \Psi e^{\alpha p^2 \tau} d\tau\right] \tag{3.3}$$

And

$$\Gamma = \left(\frac{4}{\xi h}\right)\sum_{l,m,n,=1}^{\infty} \frac{P_l(x)}{\lambda_l} \sin\left(\frac{n\pi z}{h}\right)\cos\left(\frac{m\pi y}{\xi}\right) \times e^{-\alpha p^2 t}\left[\int_0^t \Psi e^{\alpha p^2 \tau} d\tau\right] \tag{3.4}$$

And unknown function F(x, z, t) is

$$F(x,z,t)=\left(\frac{4}{\xi h}\right)\sum_{l,m,n,=1}^{\infty}\frac{P_l(x)}{\lambda_l}\sin\left(\frac{n\pi z}{h}\right)\cos\left(\frac{m\pi b}{\xi}\right)\times e^{-\alpha p^2 t}\left[\int_0^t \Psi e^{\alpha p^2 \tau}d\tau\right] \tag{3.5}$$

where

$$P_l(x)=Q\,Cos\,(\mu_m x)-W\,Sin\,(\mu_m x)$$

in which

$$Q=\mu_m\,(k_1+k_2)\cos(\mu_m h)$$

$$W=2\cos(\mu_m h)+(k_2-k_1)\,\mu_m\sin(\mu_m h)$$

$$\lambda_m^2=\int_{-h}^{h}P_l^2(x)\,dx=h\left[Q^2+W^2\right]+Sin\,\frac{(2\mu_m h)}{2\mu_m}\left[Q^2-W^2\right]$$

The eigen values μ_m are the positive roots of the characteristic equation

$$[k_1 a\cos(ah)+\sin(ah)]\,[\cos(ah)+k_2 a\sin(ah)]$$

$$=[k_2 a\cos(ah)-\sin(ah)]\,[\cos(ah)-k_1 a\sin(ah)]$$

4. DETERMINATION OF STRESS FUNCTION

Let the suitable form of χ_c satisfying (2.12) is given by

$$\chi_c=\sum_{l,m,n=1}^{\infty}\left\{y^2\left[c_1 e^{\frac{n\pi x}{a}}+c_2 e^{\frac{-n\pi x}{a}}\right]\sin\left(\frac{n\pi z}{h}\right)+y^2\left[c_3 e^{\frac{n\pi x}{a}}+c_4 e^{\frac{-n\pi x}{a}}\right]\cos\left(\frac{n\pi z}{h}\right)\right\} \tag{4.1}$$

Let the suitable form of χ_p satisfying (2.13) is given by

$$\chi_p=\left(\frac{4\alpha E a^2\xi}{h\left(a^2+\xi^2\right)}\right)\sum_{l,m,n,=1}^{\infty}\frac{P_l(x)}{\lambda_l}\sin\left(\frac{n\pi z}{h}\right)\cos\left(\frac{m\pi y}{\xi}\right)\times e^{-\alpha p^2 t}\left[\int_0^t\Psi e^{\alpha p^2\tau}d\tau\right] \tag{4.2}$$

Substituting equation (4.1) and (4.2) in (2.11), one obtains

$$\chi=\sum_{l,m,n=1}^{\infty}\left\{y^2\left[c_1 e^{\frac{n\pi x}{a}}+c_2 e^{\frac{-n\pi x}{a}}\right]\sin\left(\frac{n\pi z}{h}\right)+y^2\left[c_3 e^{\frac{n\pi x}{a}}+c_4 e^{\frac{-n\pi x}{a}}\right]\cos\left(\frac{n\pi z}{h}\right)\right\}$$

$$+\left(\frac{4\alpha Ea^2\xi}{h\left(a^2+\xi^2\right)}\right)\sum_{l,m,n,=1}^{\infty}\frac{P_l(x)}{\lambda_l}\sin\left(\frac{n\pi z}{h}\right)\cos\left(\frac{m\pi y}{\xi}\right)$$

$$\times e^{-\alpha p^2 t}\left[\int_0^t \Psi e^{\alpha p^2\tau}d\tau\right] \qquad (4.3)$$

Using (4.3) in (2.14), (2.15), (2.16) we get

$$\sigma_{xx}=\sum_{l,m,n=1}^{\infty}\left\{2\left[c_1e^{\frac{n\pi x}{a}}+c_2e^{\frac{-n\pi x}{a}}\right]\sin\left(\frac{n\pi z}{h}\right)+2\left[c_3e^{\frac{n\pi x}{a}}+c_4e^{\frac{-n\pi x}{a}}\right]\cos\left(\frac{n\pi z}{h}\right)\right\}$$

$$-\left(\frac{4\alpha Ea^2\pi^2}{h\xi\left(a^2+\xi^2\right)}\right)\left\{\sum_{l,m,n,=1}^{\infty}\frac{m^2P_l(x)}{\lambda_l}\sin\left(\frac{n\pi z}{h}\right)\cos\left(\frac{m\pi y}{\xi}\right)\times e^{-\alpha p^2 t}\left[\int_0^t\Psi e^{\alpha p^2\tau}d\tau\right]\right. \qquad (4.4)$$

$$\sigma_{yy}=\frac{\pi^2}{a^2}\sum_{l,m,n=1}^{\infty}\left\{n^2y^2\left[c_1e^{\frac{n\pi x}{a}}+c_2e^{\frac{-n\pi x}{a}}\right]\sin\left(\frac{n\pi z}{h}\right)\right.$$

$$\left.+n^2y^2\left[c_3e^{\frac{n\pi x}{a}}+c_4e^{\frac{-n\pi x}{a}}\right]\cos\left(\frac{n\pi z}{h}\right)\right\}$$

$$+\left(\frac{4\alpha Ea^2\xi}{h\left(a^2+\xi^2\right)}\right)\left\{\sum_{l,m,n,=1}^{\infty}\frac{P_l''(x)}{\lambda_l}\sin\left(\frac{n\pi z}{h}\right)\cos\left(\frac{m\pi y}{\xi}\right)\times e^{-\alpha p^2 t}\left[\int_0^t\Psi e^{\alpha p^2\tau}d\tau\right]\right. \qquad (4.5)$$

$$\sigma_{xy}=\frac{-2\pi}{a}\sum_{l,m,n=1}^{\infty}\left\{ny\left[c_1e^{\frac{n\pi x}{a}}-c_2e^{\frac{-n\pi x}{a}}\right]\sin\left(\frac{n\pi z}{h}\right)\right.$$

$$\left.+ny\left[c_3e^{\frac{n\pi x}{a}}-c_4e^{\frac{-n\pi x}{a}}\right]\cos\left(\frac{n\pi z}{h}\right)\right\}$$

$$+\left(\frac{4\pi\alpha Ea^2}{h\left(a^2+\xi^2\right)}\right)\left\{\sum_{l,m,n,=1}^{\infty}\frac{mP_l'(x)}{\lambda_l}\sin\left(\frac{m\pi y}{\xi}\right)\sin\left(\frac{m\pi z}{h}\right)\times e^{-\alpha p^2 t}\left[\int_0^t\Psi e^{\alpha p^2\tau}d\tau\right]\right. \qquad (4.6)$$

Using $\sigma_{xy}=0, \sigma_{yy}=0$, at $y=\xi$ and equation (4.5) and (4.6), we get

$$C_1=\left(\frac{-2\alpha Ea^4}{\pi^2\xi h\left(a^2+\xi^2\right)}\right)\sum_{l,m,n,=1}^{\infty}(-1)^m\left(\frac{P_l''(x)}{n^2}\right)\times\left(\frac{e^{-(\alpha p^2t+\frac{n\pi x}{a})}}{\lambda_l}\right)\left[\int_0^t\Psi e^{\alpha p^2\tau}d\tau\right] \qquad (4.7)$$

$$C_2=\left(\frac{-2\alpha Ea^3}{\pi^2\xi h\left(a^2+\xi^2\right)}\right)\sum_{l,m,n,=1}^{\infty}(-1)^m\left(\frac{P_l''(x)}{n^2}\right)\times\left(\frac{e^{-(\alpha p^2t-\frac{n\pi x}{a})}}{\lambda_l}\right)\left[\int_0^t\Psi e^{\alpha p^2\tau}d\tau\right] \qquad (4.8)$$

And $C_3=C_4=0$ (4.9)

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5. SPECIAL CASE

Set $f_1(x,y,t)=(x-a)^2(x+a)^2 y^2(y-\xi)^2 e^{-t}$, $f_2(x,y,t)=(x-a)^2(x+a)^2 y^2(y-\xi)^2 e^{h-t}$ (5.1)

6. NUMERICAL RESULTS, DISCUSSION AND REMARKS

To interpret the numerical computations, we consider material properties of **Aluminum metal**, which can be commonly used in both, wrought and cast forms. The low density of aluminum results in its extensive use in the aerospace industry, and in other transportation fields. Its resistance to corrosion leads to its use in food and chemical handling (cookware, pressure vessels, etc.) and to architectural uses.

Modulus of Elasticity, E (dynes/cm^2)	6.9×10^{11}
Thermal expansion coefficient, α_t(cm/cm-^{0}C)	25.5×10^{-6}
Thermal diffusivity, κ (cm^2/sec)	0.86
Thermal conductivity, λ (cal-cm/^{0}C/sec/ cm^2)	0.48
Length of the plate, a (m)	2
Width of the plate, ξ (m)	1.5
Width of the plate, b (m)	2
Height of the plate, h(m)	0.1

Table 1: Material properties and parameters used in this study.

7. CONCLUSION

In this paper, the temperature distribution and thermal stresses of a thin rectangular plate have been derived by using the finite Marchi-Fasulo transform and finite Fourier cosine transform and finite Fourier sine transform techniques. The results are obtained in terms of Bessel's function in the form of infinite series.

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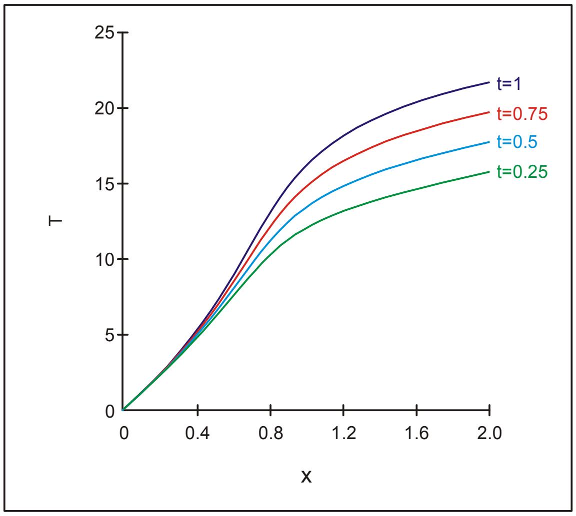
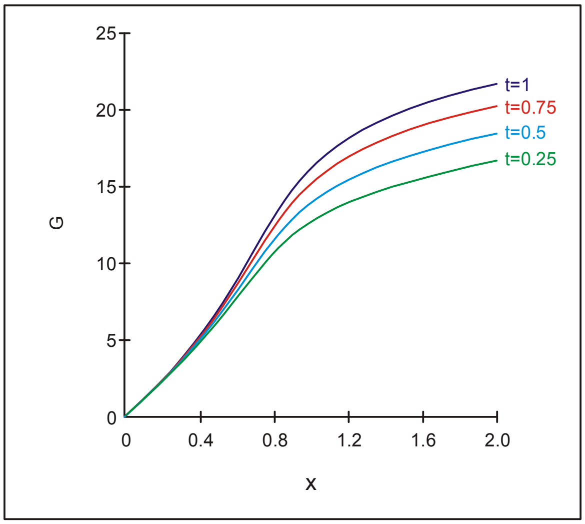


Graph 1 : Temperature distribution T versus x



Graph 2 : Unknown temperature gradient G versus x

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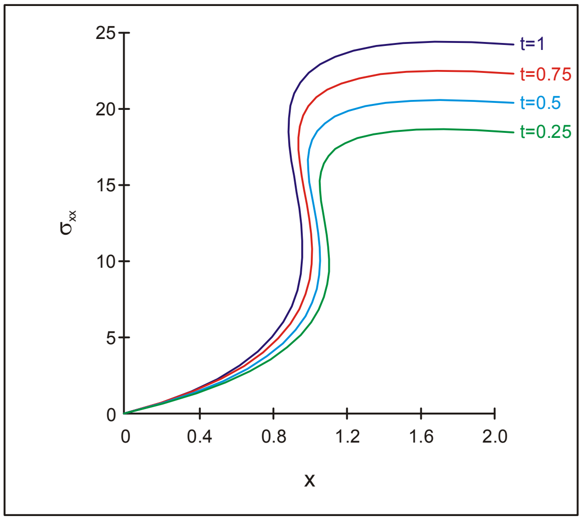
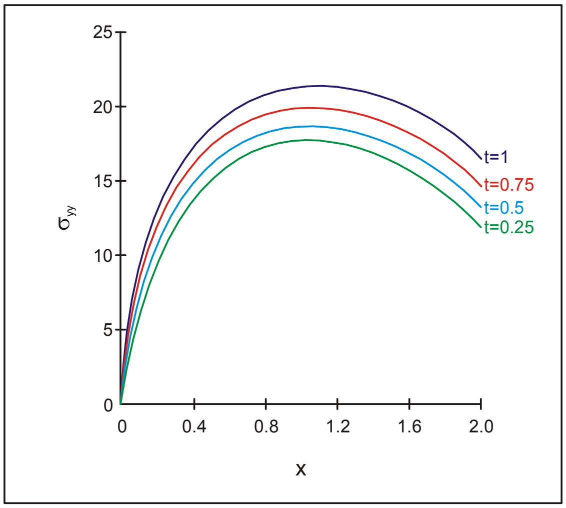


Graph 3 : Stress function σ_{xx} versus x



Graph 4 : Stress function σ_{yy} versus x

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