SOME TRIPLED FIXED POINT THEOREMS IN BIPOLAR METRIC SPACES

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ABSTRACT: In this paper, we establish the existence of unique tripled fixed point results for covariant mapping in bipolar metric spaces. Some interesting consequences of our results are achieved. Moreover, we gave an illustration which presents the applicability of the achieved results.

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1. INTRODUCTION AND PRELIMINARIES

In literature, the notions of tripled fixed points has been introduced by Berinde and Borcut [1] in 2011. Also, proved tripled fixed point theorems for contractive type mappings having mixed monotone property in partially ordered metric spaces. Furthermore, Borcut et al. [2] and Borcut [3] have introduced the concept of a tripled coincidence point for a pair of nonlinear contractive mappings. Subsequently, Karapinar [4], H. Aydi et al. [5] and K.P.R.Rao et al. ([6]-[8]) has proved some new tripled fixed point theorems. After that, Borcut et al. [9], have presented new results of the existence and uniqueness of tripled fixed point for nonlinear mappings in partially ordered complete metric spaces that extend the results in the previous works. On the other hand, Aydi et al. [10] have studied common tripled fixed point theorem for W-compatible mappings in abstract metric spaces and Murthy and Rashmi [11] have studied common tripled fixed point theorem for w-compatible mappings in ordered cone metric spaces. Very recently, Abusalim and Noorani [12] have established some new tripled coincidence point and common tripled fixed point theorems in cone metric spaces.

The notion of metric spaces has many generalizations in literature. One of the most recent of them is Bipolar metric space which is initiated by Mutlu and Gürdal [13] in 2016. Also they investigated some fixed point and coupled fixed point results on this spaces (see [13], [14]). Later, we proved some fixed point theorems in our earlier papers ([15]-[18]).

In this paper, we extend certain tripled fixed point theorem which can be considered as generalizations of literature to bipolar metric spaces. Also, we obtain some result which is related to these theorems. Finally, we give an example which presents the applicability of our obtained result.

Definition 1.1: ([13]) Let A, B be two non-empty sets. Suppose that d: A×B → [0, ∞) be a mapping satisfying the below properties:

(B_1) If d (a, b) = 0, if and only if a=b for all (a, b) ∈ A×B,

(B_2) If d (a, b) =d (b, a), for all a, b ∈ A∩B

(B_3) If $d(a_1, b_2) \leq d(a_1, b_1) + d(a_2, b_1) + d(a_2, b_2)$ for all $a_1, a_2 \in A$, and $b_1, b_2 \in B$.

Then the mapping d is termed as Bipolar-metric of the pair (A, B) and the triple (A, B, d) is termed as Bipolar-metric space.

Definition 1.2: ([13]) Assume (A_1, B_1) and (A_2, B_2) as two pairs of sets and a function as F: $A_1 \cup B_1 \to A_2 \cup B_2$ is said to be a covariant map. If F $(A_1) \subseteq A_2$ and F $(B_1) \subseteq B_2$, and denote this with F: $(A_1, B_1) \rightrightarrows (A_2, B_2,)$.

And the mapping F: $A_1 \cup B_1 \rightarrow A_2 \cup B_2$ is said to be a contravariant map. If F $(A_1) \subseteq B_2$, and F $(B_1) \subseteq A_2$, and write F: $(A_1, B_1) \rightleftarrows (A_2, B_2)$. In particular, if d_1 and d_2 are bipolar metric on (A_1, B_1) and (A_2, B_2) respectively, we sometimes use the notation F: $(A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ and F: $(A_1, B_1, d_1) \rightleftarrows (A_2, B_2, d_2)$.
Definition 1.3: *([13])* Assume (A, B, d) as a bipolar metric space. A point v ∈A ∪B is termed as a left point if v ∈ A, a right point if v ∈ B and a central point if both. Similarly, a sequence $\{a_n\}$ on the set A and a sequence $\{b_n\}$ on the set B are called a left sequence and right sequence respectively. In a bipolar metric space, sequence is the simple term for a left or right sequence. A sequence $\{v_n\}$ is considered convergent to a point v, if and only if $\{v_n\}$ is the left sequence, v is the right point and $\lim_{n\to\infty} d(v_n, v) = 0$; or $\{v_n\}$ is a right sequence, v is a left point and $\lim_{n\to\infty} d(v, v_n) = 0$. A bi-sequence $(\{a_n\}, \{b_n\})$ on (A, B, d) is a sequence on the set A ×B. If the sequence $\{a_n\}$ and $\{b_n\}$ are convergent, then the bi-sequence $(\{a_n\}, \{b_n\})$ is said to be convergent. $(\{a_n\}, \{b_n\})$ is Cauchy sequence, if $\lim_{n\to\infty} d(a_n, b_n) = 0$. In a bipolar metric space, every convergent Cauchy bi-sequence is bi-convergent. A bipolar metric space is called complete, if every Cauchy bi-sequence is convergent hence bi-convergent.

Definition 1.4: *([13])* Let (A_1, B_1, d_1) *and* (A_2, B_2, d_2) be bipolar metric spaces.

(i) A map F: $(A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ is called left-continuous at a point $a_0 \in A_1$, if for every $\epsilon>0$, there is a $\delta>0$ such that $d_1(a_0, b) < \delta$ implies that $d_2(F(a_0), F(b)) < \epsilon$ for all b∈ B_1.

(ii) A map F: $(A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ is called right-continuous at a point $b_0 \in B_1$, if for every $\epsilon>0$, there is a $\delta>0$ such that $d_1(a, b_0) < \delta$ implies $d_2(F(a), F(b_0)) < \epsilon$ for all a ∈ A_1.

(iii) A map F is considered continuous, if it left continuous at each point a∈ $A_1$and righty continuous at each point b∈ B_1.

(iv) A contravariant map F: $(A_1, B_1, d_1) \rightleftarrows (A_2, B_2, d_2)$ is continuous if and only if F: $(A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ it is continuous as a covariant map.

It is observed from the definition (1.3) that a contravariant or a covariant map F from(A_1, B_1, d_1) to (A_2, B_2, d_2) is continuous if and only if $(u_n) \rightarrow$ v on (A_1, B_1, d_1) implies $F((u_n)) \rightarrow F(v)$ on (A_2, B_2, d_2).

Definition 1.5: *([14])* Let (A, B, d) be a complete bipolar metric space, S: $(A^2, B^2) \rightrightarrows (A,B)$ be a covariant mapping. If S(a, b) = a and S(b, a) = b for (a, b)ϵ$A^2 \cup B^2$ then (a, b) is called a coupled fixed point of S.

2. RESULTS AND DISCUSSION

In this section, we give some tripled fixed point theorems for covariant mapping satisfying various contractive conditions in bipolar metric spaces.

Definition 2.1: Let (A, B, d) be a complete bipolar metric space, S: $(A^3, B^3) \rightrightarrows (A,B)$ be a covariant mapping. If S(a, b, c) = a, S(b, c, a) = b and S(c, a, b) = c for (a, b, c)ϵ$A^3 \cup B^3$ then (a, b, c) is called a tripled fixed point of S.

Theorem 2.2: Let (A, B, d) be a complete bipolar metric spaces, suppose that S: $(A^3, B^3) \rightrightarrows (A,B)$ be a covariant mapping satisfies

(2.2.1) $d(S(a, b, c), S(u, v, w)) \leq id(a, u)+jd(b, v)+kd(c, w)$ for all a, b, c ϵA and u, v, w ϵB, where i, j, k ϵ[0,1) with $i+j+k \leq h < 1$ then the mapping S: $A^3 \cup B^3 \rightarrow$ AUB has a unique fixed point of the form (a, a, a).

Proof: Let a_0, b_0, c_0 ϵA and u_0, v_0, w_0 ϵB and we construct the bisequence $(\{a_n\},\{u_n\})$, $(\{b_n\},\{v_n\})$, $(\{c_n\},\{w_n\})$ in (A,B) as

$S(a_n, b_n, c_n)=a_{n+1}$, $S(b_n, c_n, a_n)=b_{n+1}$, $S(c_n, a_n, b_n)=c_{n+1}$

$S(u_n, v_n, w_n)=u_{n+1}$, $S(v_n, w_n, u_n)=v_{n+1}$, $S(w_n, u_n, v_n)=w_{n+1}$

For n=0, 1, 2, 3,........

Now from (2.2.1), we have

$$d(a_n, u_{n+1})=d(S(a_{n-1},b_{n-1},c_{n-1}),S(u_n ,v_n, w_n)$$

$$\leq id(a_{n-1},u_n)+jd(b_{n-1},v_n)+kd(c_{n-1},w_n) \quad (2.1)$$

and

$$d(b_n, v_{n+1})=d(S(b_{n-1}, c_{n-1}, a_{n-1}),S(v_n , w_n, u_n)$$

$$\leq id(b_{n-1}, v_n)+jd(c_{n-1}, w_n)+kd(a_{n-1}, u_n) \quad (2.2)$$

and

$$d(c_n, w_{n+1})=d(S(c_{n-1}, a_{n-1}, b_{n-1}),S(w_n , u_n, v_n)$$

$$\leq id(c_{n-1}, w_n)+jd(a_{n-1}, u_n)+kd(b_{n-1},v_n) \quad (2.3)$$

for all $n \epsilon N$ and $i+j+k \leq h<1$. Let $k_n = d(a_n,u_{n+1})+ d(b_n,v_{n+1})+ d(c_n,w_{n+1})$.

For $n \epsilon N$. Combining (2.1), (2.2)and(2.3), we observe that

$$k_n = d(a_n, u_{n+1})+ d(b_n, v_{n+1})+ d(c_n, w_{n+1})$$

$$\leq (i+j+k)(d(a_{n-1}, u_n)+d(b_{n-1}, v_n)+d(c_{n-1}, w_n))$$

$$\leq hk_{n-1}$$

then we get,

$$0 \leq k_n \leq hk_{n-1} \leq h^2 k_{n-2} \leq \ldots\ldots\ldots\ldots \leq h^n k_0 \quad (2.4)$$

On the other hand

$$d(a_{n+1}, u_n) = d(S(a_n, b_n, c_n),S(u_{n-1}, v_{n-1}, w_{n-1}))$$

$$\leq id(a_n, u_{n-1})+jd(b_n, v_{n-1})+kd(c_n, w_{n-1}) \quad (2.5)$$

and

$$d(b_{n+1}, v_n) = d(S(b_n, c_n, a_n),S(v_{n-1},w_{n-1},u_{n-1}))$$

$$\leq id(b_n, v_{n-1})+jd(c_n, w_{n-1})+kd(a_n, u_{n-1}) \quad (2.6)$$

and

$$d(c_{n+1}, w_n) = d(S(c_n, a_n, b_n),S(w_{n-1}, u_{n-1}, v_{n-1}))$$

$$\leq id(c_n, w_{n-1})+jd(a_n, u_{n-1})+kd(b_n, v_{n-1}) \quad (2.7)$$

for all $n \epsilon N$ and $i+j+k \leq h<1$. Let $\lambda_n= d(a_{n+1}, u_n)+d(b_{n+1}, v_n)+d(c_{n+1}, w_n)$

for $n \epsilon N$. Combining (2.5),(2.6) and (2.7) we observe that

$$\lambda_n= d(a_{n+1}, u_n)+d(b_{n+1}, v_n)+d(c_{n+1}, w_n)$$

$\leq (i+j+k)(d(a_n, u_{n-1})+d(b_n, v_{n-1})+d(c_n, w_{n-1}))$

$\leq h\lambda_{n-1}$

then we get,

$$0 \leq \lambda_n \leq h\lambda_{n-1} \leq h^2 \lambda_{n-2} \leq \dots\dots \leq h^n \lambda_0 \quad (2.8)$$

Moreover,

$$d(a_n, u_n) = d(S(a_{n-1}, b_{n-1}, c_{n-1}), S(u_{n-1}, v_{n-1}, w_{n-1}))$$

$$\leq id(a_{n-1}, u_{n-1})+ jd(b_{n-1}, v_{n-1})+kd(c_{n-1}, w_{n-1}) \quad (2.9)$$

and

$$d(b_n, v_n) = d(S(b_{n-1}, c_{n-1}, a_{n-1}), S(v_{n-1}, w_{n-1}, a_{n-1}))$$

$$\leq id(b_{n-1}, v_{n-1})+ jd(c_{n-1}, w_{n-1})+kd(a_{n-1}, u_{n-1}) \quad (2.10)$$

and

$$d(c_n, w_n) = d(S(c_{n-1}, a_{n-1}, b_{n-1}), S(w_{n-1}, u_{n-1}, v_{n-1}))$$

$$\leq id(c_{n-1}, w_{n-1})+ jd(a_{n-1}, u_{n-1})+kd(b_{n-1}, v_{n-1}) \quad (2.11)$$

For all $n\epsilon N$ and $i+j+k \leq h<1$. Let $\mu_n = d(a_n, u_n)+d(b_n, v_n)+d(c_n, w_n)$

For $n\epsilon N$. Combining (2.9),(2.10)and (2.11), we observe that

$$\mu_n = d(a_n, u_n)+d(b_n, v_n)+d(c_n, w_n)$$

$$\leq (i+j+k)(d(a_{n-1}, u_{n-1})+ d(b_{n-1}, v_{n-1})+d(c_{n-1}, w_{n-1}))$$

$$\leq h\mu_{n-1}$$

Then we get,

$$0 \leq \mu_n\leq h\mu_{n-1}\leq h^2 \mu_{n-2}\leq\dots\dots\leq h^n \mu_0. \quad (2.12)$$

Using the property (B_3), we get

$$d(a_n, u_m) \leq d(a_n, u_{n+1})+d(a_{n+1}, u_{n+1})+\dots\dots +d(a_{m-1}, u_m)$$

$$d(b_n, v_m) \leq d(b_n, v_{n+1})+d(b_{n+1}, v_{n+1})+\dots\dots +d(b_{m-1}, v_m)$$

$$d(c_n, w_m) \leq d(c_n, w_{n+1})+d(c_{n+1}, w_{n+1})+\dots\dots +d(c_{m-1}, w_m) \quad (2.13)$$

and

$$d(a_m, u_n) \leq d(a_m, u_{m-1})+d(a_{m-1}, u_{m-1})+\dots\dots+d(a_{n+1}, u_n)$$

$$d(b_m, v_n) \leq d(b_m, v_{m-1})+d(b_{m-1}, v_{m-1})+\dots\dots+d(b_{n+1}, v_n)$$

$$d(c_m, w_n) \leq d(c_m, w_{m-1})+d(c_{m-1}, w_{m-1})+\dots\dots+d(c_{n+1}, w_n) \quad (2.14)$$

for each $n, m \epsilon N, n<m$. Then from (2.4), (2.8), (2.12), (2.13) and (2.14)

$d(a_n, u_m)+ d(b_n, v_m)+ d(c_n, w_m)$

$\leq (d(a_n, u_{n+1})+ d(b_n, v_{n+1})+ d(c_n, w_{n+1}))$

$+(d(a_{n+1}, u_{n+1})+ d(b_{n+1}, v_{n+1})+ d(c_{n+1}, w_{n+1}))$

$+................+ (d(a_{m-1}, u_{m-1})+ d(b_{m-1}, v_{m-1})+ d(c_{m-1}, w_{m-1}))$

$+(d(a_{m-1}, u_m)+ d(b_{m-1}, v_m)+ d(c_{m-1}, w_m))$

$\leq (h^n+ h^{n+1}+\ldots\ldots+h^{m-1})k_0 + (h^{n+1}+ h^{n+2}+\ldots\ldots+h^{m-1})\mu_0$

$\leq \frac{h^n}{1-h} k_0 +\frac{h^{n+1}}{1-h} \mu_0$ (2.15)

And

$d(a_m,u_n)+ d(b_m,v_m) + d(c_m, w_n) \leq (d(a_m, u_{m-1}) + d(b_m, v_{m-1}) + d(c_m, w_{m-1}))$
$+(d(a_{m-1}, u_{m-1}) + d(b_{m-1}, v_{m-1}) + d(c_{m-1}, w_{m-1}))$
$+\ldots\ldots+$
$+ (d(a_{n+1}, u_{n+1}) + d(b_{n+1}, v_{n+1}) + d(c_{n+1}, w_{n+1}))$
$+ (d(a_{n+1}, u_n) + d(b_{n+1}, v_n) + d(c_{n+1}, w_n))$
$\leq (h^n+ h^{n+1}+\ldots\ldots+h^{m-1})\lambda_0 + (h^{n+1}+ h^{n+2}+\ldots\ldots+h^{m-1})\mu_0$
$\leq \frac{h^n}{1-h} \lambda_0 +\frac{h^{n+1}}{1-h} \mu_0$ (2.16)

That is for n<m. since, for an arbitrary $\varepsilon>0$, there exist n_0 such that $\frac{h^{n_0}}{1-h} k_0 +\frac{h^{n_0+1}}{1-h} \mu_0<\varepsilon/3$ and $\frac{h^{n_0}}{1-h} \lambda_0 +\frac{h^{n_0+1}}{1-h} \mu_0<\varepsilon/3$, from (2.15) and (2.16), we have $d(a_n, u_m) + d(b_n, v_m) + d(c_n, w_m)<\varepsilon/3$ for each n, m $\geq n_0$. Then (a_n, u_n), (b_n, v_n) and (c_n, w_n) are Cauchy bisquence in (A, B). Because of completeness of (A, B, d), there exist a, b, c∈ A and u, v ,w ∈B with

$$\lim_{n\to\infty} a_n = u \qquad \lim_{n\to\infty} b_n = v \qquad \lim_{n\to\infty} c_n = w$$
$$\lim_{n\to\infty} u_n = a \qquad \lim_{n\to\infty} v_n = b \qquad \lim_{n\to\infty} w_n = c \qquad (2.17)$$

then there exist $n_1 \in N$ with $d(a_n, u) < \varepsilon/3$, $d(b_n, v) < \varepsilon/3$, $d(c_n, w) < \varepsilon/3$ and $d(a, u_n) <\varepsilon/3$, $d(b, v_n) <\varepsilon/3$, $d(c, w_n) <\varepsilon/3$ for all $n\geq n_1$ and $\varepsilon>0$. Since (a_n, u_n), (b_n, v_n), (c_n, w_n), are Cauchy bisequencess.
We get $d(a_n, u_n) < \varepsilon/3$, $d(b_n, v_n) < \varepsilon/3$, $d(c_n, w_n) < \varepsilon/3$

from (2.2.1) and (B_3), we have

$d(S(a,b,c), u) \leq d(S(a, b, c,), u_{n+1}) + d(a_{n+1}, u_{n+1}) + d(a_{n+1}, u)$
$\leq d(S(a, b, c),S(u_n, v_n, w_n)) + d(a_{n+1}, u_{n+1})+ d(a_{n+1}, u)$
$\leq id(a, u_n)+jd(b, v_n)+kd(c, w_n)+ d(a_{n+1}, u_{n+1})+ d(a_{n+1}, u)$
$\leq i\varepsilon/3 + j\varepsilon/3+ k\varepsilon/3+ \varepsilon/3+ \varepsilon/3$
$< h\varepsilon/3+2\varepsilon/3 < \varepsilon$

For each n ϵN and h<1. Then d(S(a, b, c), u)=0. Hence S(a, b, c) = u.

Similarly, we get S(b, c, a) =v and S(c, a, b)=w and S(u, v, w)=a, S(v, w, u)=b, S(w, u, v)=c.

On the other hand, from (2.17)

$d(a, u)=d\left(\lim_{n\to\infty} u_n , \lim_{n\to\infty} a_n \right)=\lim_{n\to\infty} d(a_n, u_n)=0$

and

$d(b, v)=d\left(\lim_{n\to\infty} v_n , \lim_{n\to\infty} b_n \right)=\lim_{n\to\infty} d(b_n, v_n)=0$

and

$d(c, w)=d\left(\lim_{n\to\infty} w_n , \lim_{n\to\infty} c_n \right)=\lim_{n\to\infty} d(c_n, w_n)=0.$

So, a=u, b=v, c=w. therefore, (a, b, c)$\epsilon A^3 \cap B^3$ is a tripled fixed point of S.

Now, to show the uniqueness, we begin by taking another tripled $(a^*,b^*,c^*)\ \epsilon A^3 U B^3$.

If $(a^*,b^*,c^*)\ \epsilon A^3$ then we get that

$$d(a^*,a) = d(S(a^*,b^*,c^*), S(a, b, c))$$

$$\leq id(a^*,a)+jd(b^*,b)+kd(c^*,c)$$

And

$$d(b^*,b) = d(S(b^*,c^*,a^*), S(b, c, a))$$

$$\leq id(b^*,b)+jd(c^*,c)+kd(a^*,a)$$

And $$d(c^*,c) = d(S(c^*,a^*,b^*), S(c,a,b))$$

$$\leq id(c^*,c)+jd(a^*,a)+kd(b^*,b)$$

Therefore,

$$d(a^*,a)+d(b^*,b)+d(c^*,c) \leq (i+j+k)(d(a^*,a)+d(b^*,b)+d(c^*,c))$$

$$\leq h\ (d(a^*,a)+d(b^*,b)+d(c^*,c)) \qquad (2.18)$$

Since h<1, by (2.18) this means that $d(a^*,a)=0$, $d(b^*,b)=0$, $d(c^*,c)=0$. So, we obtain that $a^*=a$, $b^*=b$, $c^*=c$.

Similarly, if $(a^*,b^*,c^*)\ \epsilon\ B^3$, we have $a^*=a$, $b^*=b$, $c^*=c$. then (a, b, c) is a unique tripled fixed point of S. finally, we show that a=b=c.

$$d(a, b) = d(S(a, b, c),S(b, c, a))$$

$$\leq id(a, b)+jd(b, c)+kd(c, a)$$

Similarly, we prove that $d(b, c) \leq id(a, b)+jd(b, c)+kd(c, a)$

and $d(c, a) \leq id(a, b)+jd(b, c)+kd(c, a)$

Therefore,

$$d(a, b)+d(b, c)+d(c, a) \leq (i+j+k)(d(a, b)+d(b, c)+d(c, a))$$

which gives a=b=c. hence (a, a, a) is tripled fixed point of S.

Corollary 1: Let (A, B, d) be a complete bipolar metric spaces, suppose that S: $(A^3,B^3) \rightrightarrows (A,B)$ be a covariant mapping satisfies

(1.1) $d(S(a, b, c), S(u, v, w)) \leq \frac{h}{2}(d(a, u)+d(b, v)+d(c, w))$ for all a, b, c ϵA and u, v, w ϵB, where h < 1. Then the mapping S: $A^3 \cup B^3 \rightarrow A \cup B$ has a unique fixed point of the form (a, a, a).

Example 2.3: Let A = $\{U_m(R)/U_m(R)$ is upper triangular matrices over R$\}$ and

B = $\{L_m(R)/L_m(R)$ is lower triangular matrices over R$\}$.

Define d: A× B→ $[0, \infty)$ as $d(P, Q) = \sum_{i,j=1}^{m} |p_{ij}-q_{ij}|$ for all $P = (p_{ij})_{m\times m} \in U_m(R)$

and Q = $(q_{ij})_{m\times m} \in L_m(R)$. Then obviously, (A, B, d) is a bipolar metric spaces.

Let the covariant maps F: $(A^3,\ B^3) \rightrightarrows$ (A, B) be defined as S (P, Q, R) = $\left(\frac{p_{ij}+q_{ij}+r_{ij}}{5}\right)_{m\times m}$

(P = $(p_{ij})_{m\times m}$, Q= $(q_{ij})_{m\times m}$, R = $(r_{ij})_{m\times m}$) $\in A^3 \cup B^3$

Consider,

$$d(S(P,Q,R), S(U,V,w)) = d\left(\left(\frac{p_{ij}+q_{ij}+r_{ij}}{5}\right)_{m\times m}, \left(\frac{u_{ij}+v_{ij}+w_{ij}}{5}\right)_{m\times m}\right)$$

$$= \sum_{i,j=1}^{m} \left|\left(\frac{p_{ij}+q_{ij}+r_{ij}}{5}\right) - \left(\frac{u_{ij}+v_{ij}+w_{ij}}{5}\right)\right|$$

$$\le \frac{1}{5}\left(\sum_{i,j=1}^{m} |p_{ij} - u_{ij}| + \sum_{i,j=1}^{m} |q_{ij} - v_{ij}| + \sum_{i,j=1}^{m} |r_{ij} - w_{ij}|\right)$$

$$\le \frac{1}{5}\left(d(SP,SU) + d(SQ,SV) + d(SR,SW)\right)$$

Therefore, the equation (1.1) is satisfied for h= $\frac{2}{5}$. Then from Corollary1, S has a unique tripled fixed point. It is obvious that $(0_{m\times m}, 0_{m\times m}, 0_{m\times m})$ is the tripled fixed point.

Theorem 2.4: Let (A, B, d) be a complete bipolar metric space, F: $(A^3, B^3) \rightrightarrows$(A, B) be a covariant mapping satisfying the following conditions

(2.4.1) $d(S(a,b,c),\ S(u,v,w)) \le \theta \max\{d(a,u), d(b,v), d(c,w)\}$

where $\theta \in (0, 1)$and a, b, $c \in A$, u, v, w$\in B$. Then the mappings S: $A^3 \cup B^3 \to A \cup B$ has unique tripled fixed point.

Proof. Let a_0, b_0, b_0 ∈A and u_0, v_0, w_0 ∈B and we construct the bi-sequences $(\{a_n\},\{u_n\})$, $(\{b_n\},\{v_n\})$, $(\{c_n\},\{w_n\})$ in (A, B) as

$S(a_n, b_n, c_n)=a_{n+1}$, $S(b_n, c_n, a_n)=b_{n+1}$, $S(c_n, a_n, b_n)=c_{n+1}$

$S(u_n, v_n, w_n)=u_{n+1}$, $S(v_n, w_n, u_n)=v_{n+1}$, $S(w_n, u_n, v_n)=w_{n+1}$

For n=0, 1, 2, 3,........

Now for all n∈ N. Let $\theta \in (0,1)$. From (2.4.1), we have

$$d(a_n, u_{n+1})=d(S(a_{n-1}, b_{n-1}, c_{n-1}), S(u_n, v_n, w_n))$$

$$\le \theta \max\{d(a_{n-1}, u_n), d(b_{n-1}, v_n), d(c_{n-1}, w_n)\} \quad (2.19)$$

and

$$d(b_n, v_{n+1})=d(S(b_{n-1}, c_{n-1}, a_{n-1}), S(v_n, w_n, u_n))$$

$$\le \theta \max\{d(b_{n-1}, v_n), d(c_{n-1}, w_n), d(a_{n-1}, u_n)\} \quad (2.20)$$

and

$$d(c_n, w_{n+1})=d(S(c_{n-1}, a_{n-1}, b_{n-1}), S(w_n, u_n, v_n))$$

$$\le \theta \max\{ d(c_{n-1}, w_n), d(a_{n-1}, u_n), d(b_{n-1}, v_n)\} \quad (2.21)$$

Combining (2.19), (2.20) and (2.21), we get

$$\max\{d(a_n, u_{n+1}), d(b_n, v_{n+1}), d(c_n, w_{n+1})\} \leq \theta\max\{d(a_{n-1}, u_n), d(b_{n-1}, v_n), d(c_{n-1}, w_n)\}. \quad (2.22)$$

Hence from (2.22), we have

$$\max\{d(a_n, u_{n+1}), d(b_n, v_{n+1}), d(c_n, w_{n+1})\} \leq \theta\max\{d(a_{n-1}, u_n), d(b_{n-1}, v_n), d(c_{n-1}, w_n)\}$$

$$\leq \theta^2\max\{d(a_{n-2}, u_{n-1}), d(b_{n-2}, v_{n-1}), d(c_{n-2}, w_{n-1})\}$$

$$\vdots$$

$$\leq \theta^n\max\{d(a_0, u_1), d(b_0, v_1), d(c_0, w_1)\}.$$

Thus

$$d(a_n, u_{n+1}) \leq \theta^n\max\{d(a_0, u_1), d(b_0, v_1), d(c_0, w_1)\}$$

$$d(b_n, v_{n+1}) \leq \theta^n\max\{d(a_0, u_1), d(b_0, v_1), d(c_0, w_1)\}$$

$$d(c_n, w_{n+1}) \leq \theta^n\max\{d(a_0, u_1), d(b_0, v_1), d(c_0, w_1)\}. \quad (2.23)$$

On other hand

$$d(a_{n+1}, u_n) = d(S(a_n, b_n, c_n), S(u_{n-1}, v_{n-1}, w_{n-1}))$$

$$\leq \theta\max\{d(a_n, u_{n-1}), d(b_n, v_{n-1}), d(c_n, w_{n-1})\} \quad (2.24)$$

and

$$d(b_{n+1}, v_n) = d(S(b_n, c_n, a_n), S(v_{n-1}, w_{n-1}, u_{n-1}))$$

$$\leq \theta\max\{d(b_n, v_{n-1}), d(c_n, w_{n-1}), d(a_n, u_{n-1})\} \quad (2.25)$$

and

$$d(c_{n+1}, w_n) = d(S(c_n, a_n, b_n), S(w_{n-1}, u_{n-1}, v_{n-1}))$$

$$\leq \theta\max\{d(c_n, w_{n-1}), d(a_n, u_{n-1}), d(b_n, v_{n-1})\} \quad (2.26)$$

Combining (2.24), (2.25) and (2.26), we get

$$\max\{d(a_{n+1}, u_n), d(b_{n+1}, v_n), d(c_{n+1}, w_n)\} \leq \theta\max\{d(a_n, u_{n-1}), d(b_n, v_{n-1}), d(c_n, w_{n-1})\}. \quad (2.27)$$

Hence from (2.27), we have

$$\max\{d(a_{n+1}, u_n), d(b_{n+1}, v_n), d(c_{n+1}, w_n)\} \leq \theta\max\{d(a_n, u_{n-1}), d(b_n, v_{n-1}), d(c_n, w_{n-1})\}$$

$$\leq \theta^2\max\{d(a_{n-1}, u_{n-2}), d(b_{n-1}, v_{n-2}), d(c_{n-1}, w_{n-2})\}$$

$$\vdots$$

$$\leq \theta^n\max\{d(a_1, u_0), d(b_1, v_0), d(c_1, w_0)\}.$$

Thus

$$d(a_{n+1}, u_n) \leq \theta^n\max\{d(a_1, u_0), d(b_1, v_0), d(c_1, w_0)\}$$

$d(b_{n+1}, v_n) \leq \theta^n \max\{d(a_1, u_0), d(b_1, v_0), d(c_1, w_0)\}$

$d(c_{n+1}, w_n) \leq \theta^n \max\{d(a_1, u_0), d(b_1, v_0), d(c_1, w_0)\}$. (2.28)

Moreover,

$d(a_n, u_n)=d(S(a_{n-1}, b_{n-1}, c_{n-1}), S(u_{n-1}, v_{n-1}, w_{n-1}))$

$\leq \theta \max\{d(a_{n-1}, u_{n-1}), d(b_{n-1}, v_{n-1}), d(c_{n-1}, w_{n-1})\}$ (2.29)

and

$d(b_n, v_n)=d(S(b_{n-1}, c_{n-1}, a_{n-1}), S(v_{n-1}, w_{n-1}, u_{n-1}))$

$\leq \theta \max\{d(b_{n-1}, v_{n-1}), d(c_{n-1}, w_{n-1}), d(a_{n-1}, u_{n-1})\}$ (2.30)

and

$d(c_n, w_n)=d(S(c_{n-1}, a_{n-1}, b_{n-1}), S(w_{n-1}, u_{n-1}, v_{n-1}))$

$\leq \theta \max\{ d(c_{n-1}, w_{n-1}), d(a_{n-1}, u_{n-1}), d(b_{n-1}, v_{n-1})\}$ (2.31)

Combining (2.29), (2.30) and (2.31), we get

$\max\{d(a_n, u_n), d(b_n, v_n), d(c_n, w_n)\} \leq \theta \max\{d(a_{n-1}, u_{n-1}), d(b_{n-1}, v_{n-1}), d(c_{n-1}, w_{n-1})\}$.(2.32)

Hence from (2.32), we have

$\max\{d(a_n, u_n), d(b_n, v_n), d(c_n, w_n)\} \leq \theta \max\{d(a_{n-1}, u_{n-1}), d(b_{n-1}, v_{n-1}), d(c_{n-1}, w_{n-1})\}$

$\leq \theta^2 \max\{d(a_{n-2}, u_{n-2}), d(b_{n-2}, v_{n-2}), d(c_{n-2}, w_{n-2})\}$

$\vdots$

$\leq \theta^n \max\{d(a_0, u_0), d(b_0, v_0), d(c_0, w_0)\}$.

Thus

$d(a_n, u_n) \leq \theta^n \max\{d(a_0, u_0), d(b_0, v_0), d(c_0, w_0)\}$

$d(b_n, v_n) \leq \theta^n \max\{d(a_0, u_0), d(b_0, v_0), d(c_0, w_0)\}$

$d(c_n, w_n) \leq \theta^n \max\{d(a_0, u_0), d(b_0, v_0), d(c_0, w_0)\}$. (2.33)

For each n, m $\in N$, n<m. Then from (2.23), (2.28), (2.31), (2.32) and (2.33)

$d(a_n, u_m)+ d(b_n, v_m)+ d(c_n, w_m)$

$\leq (d(a_n, u_{n+1})+ d(b_n, v_{n+1})+ d(c_n, w_{n+1}))$

$+(d(a_{n+1}, u_{n+1})+ d(b_{n+1}, v_{n+1})+ d(c_{n+1}, w_{n+1}))$

$+................+ (d(a_{m-1}, u_{m-1})+ d(b_{m-1}, v_{m-1})+ d(c_{m-1}, w_{m-1}))$

$+(d(a_{m-1}, u_m)+ d(b_{m-1}, v_m)+ d(c_{m-1}, w_m))$

$\leq 3 (\theta^n+ \theta^{n+1}+\ldots\ldots\ldots+\theta^{m-1}) \max\{d(a_0, u_1), d(b_0, v_1), d(c_0, w_1)\}$

$+3\,(\theta^{n+1}+\theta^{n+2}+\ldots\ldots+\theta^{m-1})\max\{d(a_0,u_0), d(b_0,v_0), d(c_0,w_0)\}.$

$\leq 3\,\frac{\theta^n}{1-\theta}\max\{d(a_0,u_1), d(b_0,v_1), d(c_0,w_1)\}$

$+3\,\frac{\theta^{n+1}}{1-\theta}\max\{d(a_0,u_0), d(b_0,v_0), d(c_0,w_0)\}\to 0$ as $n\to\infty$. (2.34)

and

$$d(a_m,u_n)+d(b_m,v_m)+d(c_m,w_n)$$
$$\leq (d(a_m,u_{m-1})+d(b_m,v_{m-1})+d(c_m,w_{m-1}))$$
$$+(d(a_{m-1},u_{m-1})+d(b_{m-1},v_{m-1})+d(c_{m-1},w_{m-1}))$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots+$$
$$+(d(a_{n+1},u_{n+1})+d(b_{n+1},v_{n+1})+d(c_{n+1},w_{n+1}))$$
$$+(d(a_{n+1},u_n)+d(b_{n+1},v_n)+d(c_{n+1},w_n))$$
$$\leq 3\,(\theta^{n}+\theta^{n+1}+\ldots\ldots+\theta^{m-1})\max\{d(a_1,u_0), d(b_1,v_0), d(c_1,w_0)\}$$
$$+3\,(\theta^{n+1}+\theta^{n+2}+\ldots\ldots+\theta^{m-1})\max\{d(a_0,u_0), d(b_0,v_0), d(c_0,w_0)\}.$$
$$\leq 3\,\frac{\theta^n}{1-\theta}\max\{d(a_1,u_0), d(b_1,v_0), d(c_1,w_0)\}$$

$+3\,\frac{\theta^{n+1}}{1-\theta}\max\{d(a_0,u_0), d(b_0,v_0), d(c_0,w_0)\}\to 0$ as $n\to\infty$. (2.35)

That is for n<m. since, for an arbitrary $\varepsilon>0$, there exist n_0 such that $\frac{\theta^{n_0}}{1-\theta}\,\eta+\frac{\theta^{n_0+1}}{1-\theta}\kappa<\varepsilon/3$ and $\frac{\theta^{n_0}}{1-\theta}\,\xi+\frac{\theta^{n_0+1}}{1-\theta}\kappa<\varepsilon/3$ where $\eta=\max\{d(a_0,u_1), d(b_0,v_1), d(c_0,w_1)\}$, $\xi=\max\{d(a_1,u_0), d(b_1,v_0), d(c_1,w_0)\}$ and $\kappa=\max\{d(a_0,u_0), d(b_0,v_0), d(c_0,w_0)\}$ from (2.34) and (2.35), we have $d(a_n,u_m)+d(b_n,v_m)+d(c_n,w_m)<\varepsilon/3$ for each $n, m\geq n_0$.

Then (a_n,u_n), (b_n,v_n) and (c_n,w_n) are Cauchy bisquence in (A, B). Because of completeness of (A, B, d), there exist a, b, c∈ A and u, v ,w ∈B with

$$\lim_{n\to\infty} a_n = u \qquad \lim_{n\to\infty} b_n = v \qquad \lim_{n\to\infty} c_n = w$$
$$\lim_{n\to\infty} u_n = a \qquad \lim_{n\to\infty} v_n = b \qquad \lim_{n\to\infty} w_n = c \qquad (2.36)$$

then there exist $n_1\in N$ with $d(a_n,u)<\varepsilon/3$, $d(b_n,v)<\varepsilon/3$, $d(c_n,w)<\varepsilon/3$ and $d(a,u_n)<\varepsilon/3$, $d(b,v_n)<\varepsilon/3$, $d(c,w_n)<\varepsilon/3$ for all $n\geq n_1$ and $\varepsilon>0$. Since (a_n,u_n), (b_n,v_n), (c_n,w_n), are Cauchy bisequencess.
We get $d(a_n,u_n)<\varepsilon/3$, $d(b_n,v_n)<\varepsilon/3$, $d(c_n,w_n)<\varepsilon/3$.

from (2.4.1) and (B_3), we have

$$d(S(a,b,c),u)\leq d(S(a,b,c,),u_{n+1})+d(a_{n+1},u_{n+1})+d(a_{n+1},u)$$
$$\leq d(S(a,b,c),S(u_n,v_n,w_n))+d(a_{n+1},u_{n+1})+d(a_{n+1},u)$$
$$\leq \theta\max\{d(a,u_n), d(b,v_n), d(c,w_n)\}+d(a_{n+1},u_{n+1})+d(a_{n+1},u)$$
$$<\theta\max\{\varepsilon/3, \varepsilon/3, \varepsilon/3\}+\varepsilon/3+\varepsilon/3$$

and

$$(S(b,c,a),v)\leq d(S(b,c,a),v_{n+1})+d(b_{n+1},v_{n+1})+d(b_{n+1},v)$$
$$\leq d(S(b,c,a),S(v_n,w_n,u_n))+d(b_{n+1},v_{n+1})+d(b_{n+1},v)$$
$$\leq \theta\max\{d(b,v_n), d(c,w_n), d(a,u_n)\}+d(b_{n+1},v_{n+1})+d(b_{n+1},v)$$
$$<\theta\max\{\varepsilon/3, \varepsilon/3, \varepsilon/3\}+\varepsilon/3+\varepsilon/3$$

and

$$(S(c,a,b),w)\leq d(S(c,a,b),w_{n+1})+d(c_{n+1},w_{n+1})+d(c_{n+1},w)$$
$$\leq d(S(c,a,b),S(w_n,u_n,v_n))+d(c_{n+1},w_{n+1})+d(c_{n+1},w)$$
$$\leq \theta\max\{d(c,wn), d(b,v_n), d(a,u_n)\}+d(c_{n+1},w_{n+1})+d(c_{n+1},w)$$
$$<\theta\max\{\varepsilon/3, \varepsilon/3, \varepsilon/3\}+\varepsilon/3+\varepsilon/3.$$

Therefore,

$$\max\{d(S(a,b,c),u), (S(b,c,a),v), (S(c,a,b),w)\}<\theta\max\{\varepsilon/3, \varepsilon/3, \varepsilon/3\}+\varepsilon/3+\varepsilon/3$$

$$<\theta\varepsilon/3+\varepsilon/3+\varepsilon/3<\varepsilon$$

For each $n\in N$ and $\theta<1$. Then $d(S(a,b,c),u)=0$, $(S(b,c,a),v)=0$ $(S(c,a,b),w)=0$. Hence $S(a,b,c)=u$. $S(b,c,a)=v$ and $S(c,a,b)=w$.

Similarly, we can show $S(u,v,w)=a$, $S(v,w,u)=b$, $S(w,u,v)=c$.

On the other hand, from (2.36)

d(a, u)=d$\left(\lim_{n\to\infty} u_n , \lim_{n\to\infty} a_n\right)$=$\lim_{n\to\infty} d(a_n, u_n)$=0

and

d(b, v)=d$\left(\lim_{n\to\infty} v_n , \lim_{n\to\infty} b_n\right)$=$\lim_{n\to\infty} d(b_n, v_n)$=0

and

d(c, w)=d$\left(\lim_{n\to\infty} w_n , \lim_{n\to\infty} c_n\right)$=$\lim_{n\to\infty} d(c_n, w_n)$=0.

So, a=u, b=v, c=w. therefore, (a, b, c)$\epsilon A^3 \cap B^3$ is a tripled fixed point of S.

Now, to show the uniqueness, we begin by taking another tripled (a^*,b^*,c^*) $\epsilon A^3 U B^3$.

If (a^*,b^*,c^*) ϵA^3 then we get that

$$d(a^*,a) = d(S(a^*,b^*,c^*), S(a, b, c))$$

$$\leq \theta \max\{d(a^*,a), d(b^*,b), d(c^*,c)\}$$

And

$$d(b^*,b) = d(S(b^*,c^*,a^*), S(b, c, a))$$

$$\leq \theta \max\{d(b^*,b), d(c^*,c), d(a^*,a)\}$$

And $$d(c^*,c) = d(S(c^*,a^*,b^*), S(c, a, b))$$

$$\leq \theta \max\{d(c^*,c), d(a^*,a), d(b^*,b)\}$$

Therefore,

$$\max\{d(a^*,a), d(b^*,b), d(c^*,c)\} \leq \theta \max\{d(c^*,c), d(a^*,a), d(b^*,b)\} \quad (2.37)$$

Since θ<1, by (2.37) this means that $d(a^*,a)=0$, $d(b^*,b)=0$, $d(c^*,c)=0$. So, we obtain that $a^*=a$, $b^*=b$, $c^*=c$.

Similarly, if (a^*,b^*,c^*) ϵ B^3, we have $a^*=a$, $b^*=b$, $c^*=c$. then (a, b, c) is a unique tripled fixed point of S.

Finally, we show that a=b=c.

$$d(a, b) = d(S(a, b, c),S(b, c, a))$$

$$\leq \theta \max\{d(a, b), d(b, c), d(c, a)\}$$

Similarly, we prove that $d(b, c) \leq \theta \max\{d(a, b), d(b, c), d(c, a)\}$

and $d(c, a) \leq \theta \max\{d(a, b), d(b, c), d(c, a)\}$

Therefore,

$$\max\{d(a, b), d(b, c), d(c, a) \leq \theta \max\{d(a, b), d(b, c), d(c, a)\}$$

which gives a=b=c. Hence (a, a, a) is tripled fixed point of S.

Example 2.5: Let A = {$U_m(R)/U_m(R)$ is upper triangular matrices over R} and

B = {$L_m(R)/L_m(R)$ is lower triangular matrices over R}.

Define d: A× B→ [0, ∞) as $d(P, Q) = \sum_{i,j=1}^{m} |p_{ij}-q_{ij}|$ for all $P = (p_{ij})_{m\times m} \in U_m(R)$

and $Q = (q_{ij})_{m\times m} \in L_m(R)$. Then obviously, (A, B, d) is a bipolar metric spaces.

Let the covariant maps $F: (A^3, B^3) \rightrightarrows (A, B)$ be defined as $F(P, Q, R) = \left(\frac{p_{ij}}{8} + \frac{q_{ij}}{4} + \frac{r_{ij}}{3}\right)_{m\times m}$

$(P = (p_{ij})_{m\times m}, Q = (q_{ij})_{m\times m}, R = (r_{ij})_{m\times m}) \in A^3 \cup B^3$

Consider,

$$d(F(P, Q, R), F(U, V, W)) = d\left(\left(\frac{p_{ij}}{8} + \frac{q_{ij}}{4} + \frac{r_{ij}}{3}\right)_{m\times m}, \left(\frac{u_{ij}}{8} + \frac{v_{ij}}{4} + \frac{w_{ij}}{3}\right)_{m\times m}\right)$$

$$= \sum_{i,j=1}^{m} \left|\left(\frac{p_{ij}}{8} + \frac{q_{ij}}{4} + \frac{r_{ij}}{3}\right) - \left(\frac{u_{ij}}{8} + \frac{v_{ij}}{4} + \frac{w_{ij}}{3}\right)\right|$$

$$\leq \frac{1}{8}\sum_{i,j=1}^{m} |p_{ij}-u_{ij}| + \frac{1}{4}\sum_{i,j=1}^{m} |q_{ij}-v_{ij}| + \frac{1}{3}\sum_{i,j=1}^{m} |r_{ij}-w_{ij}|$$

$$\leq \frac{17}{24}\max\{d(P, U), d(Q, V), d(R, W)\}.$$

Clearly S satisfies all the conditions of Theorem 2.4 and $(0_{m\times m}, 0_{m\times m}, 0_{m\times m})$ is the tripled fixed point.

Conclusion:

In the present research, we have introduced the definition of tripled fixed point and presented unique tripled fixed point results on various contractive conditions defined on bipolar metric spaces. Also gave suitable examples that supports our main results.

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